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XVI. *Index Operators.*

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SUMMARY.

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I. SYMBOLIC EXPRESSIONS AND STRUCTURAL FORMULÆ.

BOOLE'S symbolic methods have been used and extended by Heaviside and later workers. In all cases it seems to be taken for granted that the subject rests upon the calculus, and that the fundamental operator, which we shall denote by p, is the same as Boole's differentiator $\delta \equiv d/dt$. Examination will show, however, that whenever results are being interpreted p is always used consistently with Heaviside's "rule" (see 1_a), which, in effect, uses p merely as a precise indicator of a change†. Boole observed that the differential calculus deals with limits, not with differences, so that its name would be more appropriate to the calculus of finite differences than to Newton's calculus of

* Communicated by the Author.

† This view will be found developed in "Heaviside's Fractional Differentiator," Proc. Phys. Soc. xli. pp. 404–425.

limits. Boole's symbols, in addition to δ, which represents a limit, include several which denote merely a change—*e.g.*, Δ for difference, D or E for translation, and Σ for summation—in accordance with the defining equations

$$\Delta\phi(x) \equiv \phi(x+1) - \phi(x), \quad \mathrm{E}\phi(x) \equiv \phi(x+1),$$
$$\Sigma\phi(x) \equiv \phi(x-1) + \phi(x-2) + \ldots,$$

so that

$$\Delta\Sigma = 1, \quad \mathrm{E} = 1 + \Delta = e^{\delta}.$$

Boole used these symbols, with the aid of simple algebra, to specify, rather than to prove, many theorems, without any help from or reference to the calculus. The symbols are merely tools to shorten the detail work of manipulation. No assumption about them is made, and no theory of their action is needed.

It is quite otherwise with the differentiator δ, which Boole never * uses as part of an operator without referring to the calculus to justify its mode of use. Thus $e^{\delta} = \mathrm{E}$ is merely a short way of writing Taylor's theorem, on which its justification rests. The exponential expression simply denotes a definite power series in δ, and does not involve the exponential theorem. Again, when Boole establishes a form of Euler's summation formula by expanding †

$$\Sigma = \Delta^{-1} = \frac{\delta}{e^{\delta} - 1} \frac{1}{\delta}$$

as a power series in δ, with Bernoulli numbers as coefficients, he appears to be treating δ as an algebraic quantity ; but in actual fact he treats the expansion as merely suggestive, since in this, and in all such cases, he uses the "inverse form" (p. 83) "to determine a function of t such that if we perform upon it the corresponding direct operation, $(e^{\delta} - 1)$, the result will be" established. He proves the result suggested by the expansion before relying upon the latter.

This is not the method of Heaviside, who never refers to the calculus, and always uses p in algebraic formulæ just as if it were a numerical quantity. He assumes that he can use such formulæ as active operators after reducing them by any kind of analysis to some convenient form. This assumption is so violent that no one would have a good word to say for it were it not for the astonishing success

* Only his fundamental methods are here referred to. His methods are very numerous, and they are not all clear of the operator axiom.

† 'Finite Differences,' pp. 80–83 (1860).

attending its use. Heaviside was not the first, or the last, to adopt this assumption as an axiom in connexion with the differentiator δ; yet, in spite of its success, little appears to have been done, by either user or critic, to explain how it is that the use of the axiom leads in general to correct results. The employment of searching forms of analysis, such as contour integration by Bromwich, or integral equations by Carson, does not affect the fact that this assumption is made whenever the jump is taken from the algebraic formula to the operator.

Heaviside's "rule" is expressed in the first of the following:—

Definitions.

$$p^{\alpha}\frac{t^{\beta}}{\beta\,!}\equiv\frac{t^{\beta-\alpha}}{(\beta-\alpha)\,!};\quad \beta\,!\equiv\Pi(\beta)\equiv\Gamma(\beta+1)\quad . \quad . \quad . \quad . \quad 1_a$$

where α and β are any real numbers, the *index operator* is p, and p^{α} is merely an order to reduce by α the index β of any *power term*, of type $t^{\beta}/\beta\,!$, to which the operator is applied.

An *index function* of t is any group of power terms

$$f(t)\equiv\sum_r a_r t^r/r\,! \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 1_b$$

where each a_r is a constant representing the number of terms of index r, and where r may have any real value.

The generating operator, or *generator* of an index function $f(t)$, is

$$[f(t)]\equiv\sum_r a_r p^{-r}$$

or

$$[f(t)]\,.\,1=\sum_r a_r p^{-r}\,.\,\frac{t^0}{0\,!}=f(t)\quad . \quad . \quad . \quad . \quad 1_c$$

An index function f can be looked upon as a family group of power terms. the ages or states of development of which are given by the corresponding indices; while f and $p^{\alpha}f$ represent the same family group at two different times, the age difference of any pair of terms in f being the same as that of the corresponding pair in $p^{\alpha}f$.

If $\phi(p)$ is any power series in p, and if $f(t)$ is any index function of t, a perfectly definite meaning attaches to each of the expressions

$$\phi(p)f(t)\equiv\phi(p)\,[f(t)]\,.\,1,$$

and each can be regarded as a structural formula.

It will be clear that products involving p follow the laws of algebraic combination, and that fractional * powers of p can be used. This is not the case with Boole's differentiator δ, owing to the difficulty about constants both on integration and on differentiation. It has been pointed out in several papers that δ is not a commutative operator.

Such a definition of p makes it a descriptive symbol like the Δ, E, and Σ used by Boole, and one having the same general characteristics. It can be used with ordinary algebra to shorten manipulation without needing any theory to explain its action. An expression in p can be used to state a formula in a way which, in general, is preferable to that of giving to the formula a special name or letter, since it is structural, not arbitrary, and makes no call upon the memory. "Granting the desirability of having special short ways of representing important functions," the operators "may be themselves regarded as the special short ways, shorter than the other ways in fact." Heaviside shows many such "short ways," but he regards the operator as being much more than a structural formula for a series. He looks upon it as being the function itself, as distinct from any formula got from it, which is "only the dress, and not always a convenient fit," no dress being suitable for all occasions. This view is far-reaching and much harder to justify. It rests upon the operator axiom.

In illustration of descriptive symbols we may quote some formulæ which we shall find useful later on, and which are given by Knopp † in connexion with Bernoulli numbers and polynomials :

$$\frac{t}{e^t-1}=e^{\mathrm{B}t}, \qquad 2_a$$

$$(1+\mathrm{B})^n=\mathrm{B}^n \quad \text{for} \quad n \geq 2, \qquad 2_b$$

$$t\frac{e^{mt}-1}{e^t-1}=\sum_{n=0}^{\infty}\frac{(m+\mathrm{B})^n-\mathrm{B}^n}{n\,!}t^n \qquad 2_c$$

Each of these formulæ becomes clear with the convention "expand in powers of B, and change B^r to B_r." The notation is such that B_r is zero for every odd integer r except unity. No dubious assumption is involved in these

* Heaviside was not the first to use such fractional powers ; a case occurs in Boole's 'Differential Equations,' 4th ed. p. 401, but it is classed under "forms purely symbolical"—*i. e.*, as a form mathematically suggestive, but one whose full explanation is indefinitely postponed.

† 'Theory and Application of Infinite Series,' Blackie, English translation, p. 183 &c.

expressions, but if, in order to deduce 2_c from 2_a, we use the intermediate step

$$e^{Bt}(e^{mt}-1)=e^{(B+m)t}-e^{Bt},$$

we arrive at the correct result by assuming the operator axiom, since we treat a symbolic expression in B as if B were an algebraic quantity. The manipulation needed in this case to justify the step is so simple that it may be regarded by many as obvious, but, if so, it is because the manipulation can be mentally pictured, and not because it is unnecessary.

We may next illustrate the purely descriptive use of p, as a means of shortening the precise expression of a result in the proof of which it makes no pretence to share.

Consider an Euler integral of the second kind,

$$\frac{1}{p^{n+1}}=\int_0^\infty e^{-pt}\frac{t^n}{n\,!}\,dt,$$

in which p is a positive number. We have a result consistent with

$$p\cdot\frac{1}{p^{n+1}}\cdot 1=\frac{t^n}{n\,!},$$

in which p is Heaviside's index operator 1_a.

Similarly if, following Carson, we take the integral equation

$$F_s(p)=\int_0^\infty f_s(t)e^{-pt}\,dt \quad . \quad . \quad . \quad . \quad . \quad 3_a$$

for any case in which $f_s(t)$ is an index function 1_b, we can, by substituting in the integrand, obtain a number of Euler integrals yielding on superposition a result consistent with

$$p\cdot F_s(p)\cdot 1=f_s(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 3_b$$

where, as before, p is a number in the former equation and the index operator in the latter.

Moreover, if we have three integral equations like 3_a distinguished by suffixes $s=1$, 2, or 3, and if also we have the relation

$$F_1(p)=F_2(p)\cdot F_3(p),$$

we get Borel's theorem (for index functions)

$$f_1(t)=\int_0^t f_2(x)f_3(t-x)\,dx,$$

since the integrand, on expansion, consists of a number of terms typified by

$$k_{mn}x^m(t-x)^n,$$

corresponding with an Euler integral of the first kind. On using this we get a power series in t which, with the aid of the relations between the coefficients given by the equation between the $F(p)$ functions, will be found to be $f_1(t)$.

The Borel theorem for index functions is thus seen to be a direct result of the manipulation of Euler's two integrals, using p throughout simply as a number. For our present purpose the important point to note is that the results are all consistent with three operator equations like 3_b, and also that we can express Borel's theorem operationally as follows:—

If
$$p[f_1(t)] = [f_2(t)][f_3(t)] \quad . \quad . \quad . \quad . \quad . \quad 4_a$$

then
$$f_1(t) = \int_0^t f_2(x) f_3(t-x)\,dx, \quad . \quad . \quad . \quad 4_b$$

where the former equation gives the relation which must hold between the generators 1_c of the functions satisfying the latter equation.

These formulæ are established as deductions from old theorems by simple manipulation without the use of operators or of the operator axiom; but each result can be expressed as a function of the operator p and as a structural formula for the corresponding function of t.

Now, if close examination be made of the old work of Heaviside, and also of the recent work of Carson * and Balth v. d. Pol, it will be found that a considerable portion of the theorems and examples given rest on proofs which, with a slight change of wording, only involve the use of p in its descriptive aspect as an index changer 1_a. This is particularly the case when the functions dealt with are of the index type. The restriction of analysis to index functions is not so great as at first it appears to be. Nearly all the functions used in physics are of this type. In the papers cited nearly all the theorems and examples refer to index

* Carson, 'Electric Circuit Theory and the Operational Calculus,' McGraw-Hill, New York; Balth v. d. Pol, "Operational Solution of Linear Differential Equations," Phil. Mag. viii. (Dec. 1929). The former paper, pp. 39–40, gives a list of fifteen integral equations for use with operators, and the latter paper, pp. 894–896, gives a list of over thirty operational results. In every case only an index function is involved.

functions, and, though closed expressions involving p are sometimes used, they are interpreted as the corresponding expansions in powers of p.

The examples illustrate in general a machine-like way of manipulating infinite series, with the result of establishing, easily and rapidly, relationships between index functions. The index operator seems to be the natural tool to aid such processes, while the interpretation of p as such detracts in no way from the power and convenience of the method as shown in the papers referred to.

All cases in which use is made of the operator axiom must be excepted from the above statement, and it is also necessary to explain cases in which use is made of Heaviside's "unit function" $\mathrm{H}(t)$, which is "unity for positive, and zero for negative, values of t."

In Heaviside's work the operator is taken from the differential equation. The $\mathrm{H}(t)$ function is stated or implied in almost every example, and is often used in conjunction with a form of Fourier's theorem. It is, in general, so much involved in his work that it is often assumed that its use is essential to his method and that this method is based upon the calculus.

The view here advanced is that the $\mathrm{H}(t)$ function is needed in order correctly to formulate the mathematics of the problem; that the method, though usable with the calculus, is not dependent upon it; and that the use of $\mathrm{H}(t)$, though not essential to the method itself, may possibly be needed to justify the use of the operator axiom in cases otherwise doubtful.

In regard to the last point, every dynamic problem involves something analogous to the application of a force $f(t)$ at and from $t=0$, so that the variable can only have positive values. This restriction can be mathematically expressed by calling the force $f(t) \times \mathrm{H}(t)$. Now, by Laurent's theorem, if a function of t is analytic for a range $t_1 < |t| < t_2$, it can be regarded as an index function for such range. The use of $\mathrm{H}(t)$ in analysis restricts the use of the functions dealt with to a limited range of values of the variable, and appears to make it legitimate to regard these functions, for the purpose of the problem, as index functions, in connexion with which the use of the operator axiom is either not needed or is comparatively easy to justify. Before, however, discussing the operator axiom it is desirable to clear the ground by showing that the $\mathrm{H}(t)$ function is definite, that it can be treated as any ordinary index function, and that it works with precision in connexion with Fourier series and integrals.

II. The Heaviside Function in Use with Fourier Series and Integrals.

The Heaviside function may be defined * as follows :—

$$H(t)=\frac{1}{2}+\frac{1}{\pi}\operatorname{Si}(nct), \quad . \quad . \quad . \quad . \quad . \quad 5$$

where

$$\operatorname{Si}(nct)\equiv\int_0^{nct}\frac{\sin z}{z}\,dz, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 5_a$$

$$nc.c^{\nu}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 5_b$$

n, c, and ν are fixed positive numbers ; n is an infinite integer, c is an infinitesimal, and ν is an infinite number. There is no limit to the smallness of c or to the largeness of ν, but, once chosen, these numbers must be considered constant throughout analysis. . 5_c

As thus defined $H(t)$ is an index function of t with all indices positive and integral. It is an infinite series which, for all values of t, is even more convergent than the corresponding series for $\sin(nct)$. In order to show its working in conjunction with Fourier formulæ we propose to consider an old theorem—Euler's summation formula—in connexion with a published proof involving Fourier analysis, and to translate this proof † into the language of the $H(t)$ function.

The proof naturally consists of two parts—a preliminary portion involving the collection of known results, and a final part forming the proof proper.

The first part establishes a series of functions $P_1(x)$, $P_2(x)$, etc., each of which has the following properties :—

It is periodic with wave-length unity . . . . 6_a

It is alternating, in the electrotechnical sense, so that its mean value, taken over any complete wave-length, is zero . . . . 6_b

* For further details see "Impulse Functions," Phil. Mag. [7] xi. p. 345 (Feb. 1931). In that paper an additional factor $t^c/c!$ is used with $H(t)$, but this factor is here omitted in order to bring out the exact correspondence between $H(t)$ and Fourier series and integrals. If

$$H_c(t)\equiv H(t)\times t^c/c!,$$

the use of $H_c(t)$ makes negligible the fluctuations of the Gibbs phenomenon by reducing them all to infinitesimals of order c^{ν}. This can be illustrated by the figure representing (7_c) below. If $\operatorname{St}_c(x)$ is the modified form of $\operatorname{St}(x)$, the steps of the staircase are all rounded at the corners, with each tread quite flat right up to the next integral value of x.

† The proof selected will be found in Knopp, *loc. cit.* pp. 520–526.

It is the derivative of the next function in the series

$$P_r(x) = p \cdot P_{r+1}(x), \quad p \equiv d/dx \quad . \quad . \quad . \quad . \quad 6_c$$

For any positive integer m

$$P_1(o) = P_1(m) = 0,$$

and

$$P_r(o) = P_r(m) = B_r/r\,! \quad \text{for} \quad r \gtrless 2 \quad . \quad . \quad . \quad 6_d$$

The formula for $P_r(x)$ as given by Knopp is

$$P_{2\lambda+1}(x) = \sum_{s=1}^{\infty} \frac{2 \sin 2\pi s x}{(-2\pi s)^{\lambda+1}}, \quad . \quad . \quad . \quad . \quad . \quad 6_e$$

with a corresponding formula for its derivative $P_{2\lambda}(x)$, using cos instead of sin.

We thus have

$$P_1(x) = -\sum_{s=1}^{n_1} \frac{\sin 2\pi s x}{\pi s}, \quad n_1 = \infty \quad . \quad . \quad 7_a$$

This is the Fourier series got in the ordinary way for $y_1 = x - \frac{1}{2}$, for $0 < x < 1$, on the assumption that y_1 can be so expanded. The assumption is not true at the limits, and indeed, unless infinitesimals are neglected, it is not true even within the limits. In text-books the series is not summed by a Fourier integral but by trigonometrical methods. No consideration is given to the value of P_1 for infinitesimal values of x, except where the Gibbs phenomenon is referred to, and P_1 is alleged to be discontinuous * in the close neighbourhood of integral values of x, though it is admitted that for any definite value of n, however large, P_1 is a perfectly continuous function. No difficulty arises about discontinuity if we take the simple view that a quantity F involving x, r, and n, has a value, when $x \to 0$, $r \to a$, and $n \to \infty$, which depends not only on the infinitesimal x but also on the infinitesimals $(r-a)$ and $1/n$, and that, in general, it is possible to find a precise value for F only when each of these infinitesimals can be expressed definitely in terms of some common unit.

Now it is just when x is infinitesimal that it is possible to find P_1 by means of a Fourier integral, because for $n_1 = \infty$

$$-P_1(x) = \frac{1}{\pi} \sum_{s=1}^{n_1} \frac{\sin 2\pi s x}{2\pi s x} 2\pi x = \frac{1}{\pi} \int_{2\pi x}^{2\pi n_1 x} \frac{\sin z}{z} dz,$$

* This conclusion appears to be due to the vague way in which an infinite number, n, is often regarded. All values of x seem ignored for which $0 < nx < \infty$.

when, and only when, x is infinitesimal, since we can in such case put

$$z=2\pi sx, \qquad dz=2\pi x.$$

We thus get

$$-P_1(x)=\frac{1}{\pi}\operatorname{Si}(2\pi n_1 x)=\frac{1}{\pi}\operatorname{Si}(\lambda)$$

for any finite value of λ for which

$$x=\frac{\lambda}{2\pi n_1},$$

and for such infinitesimal values of x we have the finite fluctuations of the Gibbs "phenomenon."

Now we can put

$$2\pi n_1=nc=c^{-\nu},$$

with the meanings of n, c, ν given in 5_c.

Thus we have proved for infinitesimal, but not for finite, values of x that, using definition 5,

$$P_1(x)=-\frac{1}{\pi}\operatorname{Si}(ncx)=-\tfrac{1}{2}+1-H(x).$$

We have, however, in

$$x-\frac{1}{\pi}\operatorname{Si}(ncx)$$

a continuous function of x, which is zero at $x=0$, and changes for $\frac{1}{2}-c$ at $x=-c$ to $c-\frac{1}{2}$ at $x=+c$; while between $x=+c$ and $x=1-c$ it is $x-\frac{1}{2}$. It has the same value at $x=-c$ as at $x=1-c$. It is therefore that portion of the periodic function $P_1(x)$ which lies between the limits $x=-c$ and $x=1-c$. If we use B_1 for $-\frac{1}{2}$, we have from 5 within the stated limits of x

$$P_1(x)=x-\frac{1}{\pi}\operatorname{Si}(ncx)=\frac{x+B_1}{1}+1-H(x). \quad . \quad . \quad 7_b$$

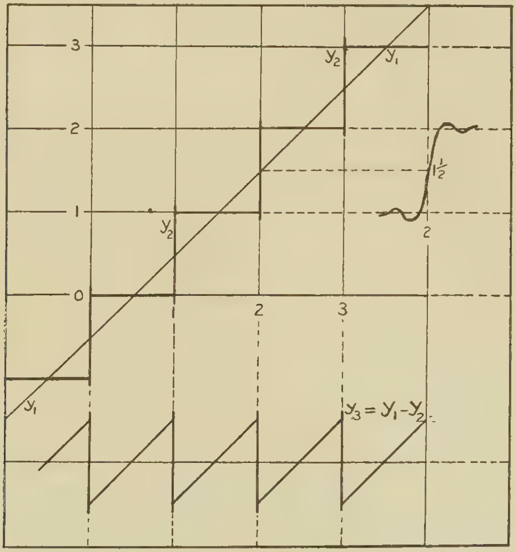
Now consider the "step" or "staircase" function $St.(x)$ defined by

$$\begin{aligned}St.(x)\equiv H(x)+H(x-1)+H(x-2)+\ldots\\ -[1+H(-x-1)+H(-x-2)+\ldots] \quad . \quad . \quad 7_c\end{aligned}$$

This will be seen to denote a staircase with utit treads and risers, as in the figure, with the tread for $0<x<1$ at the level $y=0$. The Gibbs fluctuations, at the integral values of x, will be so close together that they can only be shown graphically by vertical extensions of the risers. If, however,

the scale of x be magnified immensely near some integral value (say 2) of x, the corresponding riser, shown in the figure, passes steeply through the point $(2, 1\frac{1}{2})$, and fluctuates about $y=1$ for $x>2$ and about $y=2$ for $x>2$. The line $y_1=x-\frac{1}{2}$ passes through the mid-point of each riser. If a new curve $y_3=y_1-y_2$ be formed, where $y_2=\mathrm{St}(x)$, we get the periodic curve $P_1(x)$ shown at the bottom of the figure,

$$y_3=y_1-y_2=x-\tfrac{1}{2}-\mathrm{St}(x),$$



$y_1=x-\frac{1}{2}$. $y_2=\mathrm{S}t(x)$. $y_3=y_1-y_2$.

with vertical Gibbs fluctuations at the angular points. Thus we have, for all values of x,

$$P_1(x)=\frac{x+B_1}{1}-\mathrm{St}(x) \quad . \quad . \quad . \quad . \quad . \quad 7a$$

$P_1(x)$, as so defined, is alternating as well as periodic, since its mean value is zero for any unit stretch of x. Its integral $P_2(x)$ will thus be periodic whatever the constant of integration, while for one, but only for one, value of the latter it will also be alternating. Assuming this constant used, by integrating $P_2(x)$ and by adding a suitable constant we can get $P_3(x)$ as an alternating and periodic quantity. By continuing this process we obtain for any positive integer r

$$P_{r+1}(x)\equiv\frac{(x+B)^{r+1}}{(r+1)\,!}-p^{-r}\mathrm{St.}(x), \quad . \quad . \quad . \quad 8$$

where the first expression on the right denotes symbolically a special polynomial (see 2_c), and where p^{-r} denotes the operation, carried out r times in succession, of integration with regard to x from 0 to x, such notation being consistent with Heaviside's "rule" when applied to any integrand which is an index function with all its indices above (-1).

In order to prove 8 we note from the figure that, for the range $0<x<1$, the value of $p^{-r}\mathrm{St}(x)$ is zero if we neglect fluctuations whose integral value can be shown to be an infinitesimal of order c^{ν}. It is thus easy to establish 8 by induction for this range, since, if

$$\mathrm{P}_r(x)=\frac{(x+\mathrm{B})^r}{r\,!}$$

$$\mathrm{P}_{r+1}(x)=\int_0^x \mathrm{P}_r(x)\,dx+\frac{\mathrm{K}_r}{(r+1)\,!}=\frac{(x+\mathrm{B})^{r+1}-\mathrm{B}^{r+1}+\mathrm{K}_r}{(r+1)\,!}$$

where $\mathrm{K}_r/(r+1)\,!$ is the constant of integration to be so chosen that

$$\int_0^1 \mathrm{P}_{r+1}(x)\,dx=0,$$

and it will readily be seen that this result is secured by making $\mathrm{K}_r=\mathrm{B}_{r+1}$. Thus, for the range $0<x<1$, since 8 is true for $r=0$, it must be true for all values of r. Moreover 8 must be true for all values of x, since its form shows that P_{r+1} is an integral of P_r, while we have just shown that the correct constant of integration has been added to make each P function both alternating and periodic. It is, however, not directly evident from 8 that if $x=m+\xi$, where $0<\xi<1$,

$$\mathrm{P}_{r+1}(m+\xi)=\frac{(\xi+\mathrm{B})^{r+1}}{(r+1)\,!} \quad . \quad . \quad . \quad . \quad . \quad 9$$

for every positive integer m. To verify this we must find $p^{-r}\mathrm{St}(x)$ for the range $0<x<m+1$. To do this, consider

$$\int_0^x \frac{(x-s)^k}{k\,!}\mathrm{H}(x-s)\,dx\equiv\int_{-s}^{x-s}\frac{y^k}{k\,!}\mathrm{H}(y)\,dy$$

for any two positive integers s and k. The presence of the H factor in the integrand renders the latter non-existent for certain values of the variable, and leaves it unchanged for the rest; so that it simply alters the limits of the integral, which yields the result

$$\frac{(x-s)^{k+1}}{(k+1)\,!}$$

instead of one denoted by this term as diminished by its special value for $x=0$. But this is not all. The integral itself does not exist unless $x>s$, so that to represent it precisely* we must multiply the above expression by $\mathrm{H}(x-s)$. Thus by successive integration we get

$$p^{-r}\frac{(x-s)^k}{k!}\mathrm{H}(x-s)=\frac{(x-s)^{k+r}}{(k+r)!}\mathrm{H}(x-s)$$

for all positive integers k and r from zero upwards in each case.

Now for the range $0<x<m+1$ it follows from 7_c that it is necessary and sufficient to put

$$\mathrm{St}(x)=\mathrm{H}(x)+\mathrm{H}(x-1)+\ldots+\mathrm{H}(x-m)-1,$$

whence

$$p^{-r}\mathrm{St}(x)=\frac{x^r}{r!}\mathrm{H}(x)+\sum_{s=1}^{m}\frac{(x-s)^r}{r!}\mathrm{H}(x-s)-\frac{x^r}{r!}.$$

If $x>m$, the first and last terms on the right cancel, and for the rest $\mathrm{H}(x-s)=1$; thus

$$p^{-r}\mathrm{St}(x)=\sum_{s=1}^{m}\frac{(x-s)^r}{r!}=\frac{(\xi+m-1)^r+\ldots+\xi^r}{r!}$$

$$=\frac{(\xi+m+\mathrm{B})^{r+1}-(\xi+\mathrm{B})^{r+1}}{(r+1)!} \quad . \quad . \quad . \quad 10$$

by a known formula†.

If now we use 10 with 8 we get 9. We can reach the same result operationally as follows (using 2_c) :—

$$\mathrm{St}(x)+1-\mathrm{H}(x)=\sum_{s=1}^{m}\mathrm{H}(x-s)=(e^{-p}+e^{-2p}+\ldots+e^{-mp})\mathrm{H}(x)$$

$$=\frac{e^{-mp}}{p}p\cdot\frac{e^{mp}-1}{e^{p}-1}\mathrm{H}(x)$$

$$=\frac{e^{-mp}}{p}\sum_{s=1}^{\infty}\frac{(m+\mathrm{B})^s-\mathrm{B}}{s!}p^s\mathrm{H}(x).$$

Now if we operate throughout by p^{-r} the term

$$p^{-r}[1-\mathrm{H}(x)]=\frac{x^r}{r!}-\frac{x^r}{r!}\mathrm{H}(x)=0$$

* The infinitesimals are ignored, but they will be found negligible if integration by parts is applied to the above integral. The new integral involving the derivative of $\mathrm{H}(x-s)$ will be impulsive and negligible in value.

† Bromwich, 'Infinite Series,' 2nd ed. pp. 300–301.

for $0 < x$; hence we get

$$p^{-r}\mathrm{St}(x) = e^{-mp}\sum_{s=1}^{m}\frac{(m+\mathrm{B})^s-\mathrm{B}^s}{s\,!}\,\frac{x^{r+1-s}}{(r+1-s)\,!}\mathrm{H}(x)$$

$$= e^{-mp}\frac{(x+m+\mathrm{B})^{r+1}-(x+\mathrm{B})^{r+1}}{(r+1)\,!}\mathrm{H}(x)$$

$$= \frac{(\xi+m+\mathrm{B})^{r+1}-(\xi+\mathrm{B})^{r+1}}{(r+1)\,!}\mathrm{H}(x-m),$$

and this reproduces 10, since $\mathrm{H}(x-m) = 1$ if $x > m$. In the above we have used Knopp's symbolic expressions 2 in some cases algebraically, but it will be found that the equivalences stated can readily be justified.

It will thus be clear that the properties 6_a to 6_d of the P functions are as true when these functions are expressed in terms of $\mathrm{H}(x)$, as in 8, as when given in terms of Fourier series, as in 6_e. They are the same functions expressed in different language.

A similar statement holds good for the proof of Euler's summation formula as given in terms of these P functions by Knopp [who points out that their use for this purpose is due to Wirtinger].

Thus, if $f(x)$ be any function of x which, with each of its derivatives, is finite for all values of x from 0 to some positive integer n, and if we define

$$\mathrm{S}_n \equiv f(0) + f(1) + \ldots + f(n),$$

$$\mathrm{F}_n \equiv \int_0^n f(x)\,dx, \quad f_r(s) \equiv \left[\frac{d^r}{dx^r}f(x)\right]_{x=s},$$

$$\mathrm{I} = \int_0^n \mathrm{P}_1(x) f_1(x)\,dx,$$

we can get, on integration by parts, two equal values, I_1 and I_2, for I.

On integrating P first, and using the relations 6, we obtain, exactly as in Knopp's proof,

$$\mathrm{I}_1 = \sum_{r=2}^{\lambda}(-1)^r\frac{\mathrm{B}r}{r!}\left[f_{r-1}(n) - f_{r-1}(0)\right]$$
$$+ (-1)^\lambda\int_0^n \mathrm{P}_\lambda(x) f_\lambda(x)\,dx.$$

On integrating $f_1(x)$ first we get

$$\mathrm{I}_2 = \Big[\mathrm{P}_1(x) f(x)\Big]_0^n - \int_0^n f(x)\frac{d\,\mathrm{P}_1(x)}{dx}\,dx.$$

By using (6 d) and (7 d) this reduces to

$$I_2 = -\int_0^n f(x)[1-p.\mathrm{St}.(x)]\,dx = -F_n + \int_0^n f(x)p.\mathrm{St}(x)\,dx.$$

Now from (7 c), if $0 \leq x \leq n$, we have

$$\mathrm{St}(x) = -1 + H(x) + H(x-1) + \ldots + H(x-n),$$

or

$$p.\mathrm{St}(x) = \sum_{r=0}^{n} pH(x-r).$$

By Fourier's theorem, in its impulse form, we have

$$\begin{aligned}\int_0^n f(x)pH(x-r)\,dx& \\ &= f(r) \quad \text{if} \quad 0<r<n \\ &= \tfrac{1}{2}f(r) \quad \text{if} \quad r=0, \text{ or } n,\end{aligned}$$

whence

$$\begin{aligned}I_2 &= -F_n + \tfrac{1}{2}f(0) + f(1) + \ldots + f(n-1) + \tfrac{1}{2}f(n) \\ &= -F_n + S_n - \tfrac{1}{2}[f(0) + f(n)],\end{aligned}$$

so that by equating I_1 to I_2 we get Euler's summation formula.

The contention is not that the two proofs are different, but that they are the same except for language. They are also essentially of equal length, allowance being made for the fact that the preliminary part is in one case assumed from known results, while in the other it has necessarily to be established in the new form.

The example will suffice to show that so far as Heaviside's operator method is concerned $H(x)$ can be looked upon as an ordinary index function, and that its use is not essential to the working of the method.

III. The Operator Axiom.

The differential equation expressing an unknown function $F(t)$ of t in terms of a known function $f(t)$ yields an operator equation which may be written

$$F(t) = \phi(p)f(t).$$

The operator axiom asserts that we can convert $\phi(p)$, by any mode of analysis treating p as an algebraic quantity, to some more convenient form $\phi'(p)$, and that we can use the latter in place of the former as the working operator.

This must be true if $\phi(p) \equiv \phi'(p)$ is an algebraic identity, provided that the operator p satisfies the laws of algebraic combination.

The simplest and most frequent case arises from a differential equation

$$D(p)F(t) = N(p)f(t),$$

where N and D are two polynomials in p. Its solution, as made known by the work of Boole, depends on expressing N/D as the sum of partial fractions of type $A/(p-a)$, where a is a root of $D(p)$ and A is the corresponding constant. The sum of these fractions forms with N/D an algebraic identity in p.

Heaviside took Boole's * solution and improved it by using a simple way of expressing the constants. The result is known as his expansion theorem, one which has been much used in electrotechnics.

Boole's solution made use of the calculus; but without any such aid it seems legitimate to use in place of N/D the identical function of p, and to express F as the sum of a number of partial solutions of type

$$F_a = (p-a)^{-1} f(t), \quad \text{where} \quad (p-a)[F_a] = [f(t)];$$

but to go further needs the use either of the calculus or of some identity method. Thus, if $[f(t)]$ is a given power series in p, we can assume another for $[F_a]$ having an infinite number of terms with unknown coefficients, and can find the latter by using the equation to balance terms having like powers of p. This introduces an arbitrary constant yielding the full calculus solution. There will, however, be an unbalanced end term of type $p^{\pm w}$ corresponding with $t^w/w!$, where w is an infinite integer. This term will vanish as a numerical quantity whether w be positive or negative. But though this remainder term can be neglected if F_a is wanted as a

* For Boole's solution see 'Differential Equations,' p. 391 (1877). Boole gracefully acknowledged prior publication by Lobatto, Amsterdam, 1837 (see 'Finite Differences,' p. 108 (1860)). Heaviside refers to Boole's solution in 'Electrical Papers,' ii. p. 226. In technical publications mystery has been made of the origin of the expansion theorem, since Heaviside in 'Electromagnetic Theory' stated it without proof or reference, and also without making any personal claim; but he did not bother about such matters, and seemed to regard them all, even the first, as waste of time. In place of proof he relied on trial and test. There is, however, an important difference between Boole's solution and that of Heaviside. The former is a delicate piece of apparatus, which for each case must be set up with some skill and care; the latter is a robust machine, or measuring instrument, easy to work and always ready for use.

numerical quantity, it may be quite unjustifiable to do so if $[F_a]$ is wanted either as an operator or as an operand.

Now Carson treats the integral equation as if available not only when p is a simple or complex number, but also when it is the operator. Bromwich, when he interprets an operator as a complex integral, appears to make the same assumption.

Let us assume that the Carson integral

$$F(p)=\int_0^\infty e^{-px}f(x)\,dx \quad . \quad . \quad . \quad . \quad . \quad 11_a$$

is an identity in p, and that p can be used as the index operator, the variable being t. Apply each element of the equation to the operand $p.H(t)$; we get

$$pF(p)H(t)=\int_0^\infty f(x)pe^{-px}H(t)\,dx \quad . \quad . \quad . \quad 11_b$$

Now pe^{-px} is a power series in p with positive integral indices, while $H(t)$ is an index function in t with like indices. In such cases the index operator can be replaced by Boole's differentiator, so that the second element of 11_b becomes

$$\int_0^\infty f(x)\frac{dH(t-x)}{dt}\,dx,$$

and this by Fourier's theorem in its impulse form is indistinguishable from $f(t)$ except at a discontinuity at which there are Gibbs fluctuations whose measure is always proportional to the impulsive change at the discontinuity (finite, or infinite, as the case may be). If, therefore, $f(t)$ is continuous for the range $t_1<t<t_2$, we have for every such value of t

$$pF(p)H(t)=f(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad 11_c$$

where p is the index operator. We have not assumed that $f(t)$ is an index function, as in the previous case 3_b, so that $F(p)$ may prove to be a closed expression in p. Its interpretation as an operator may not be obvious, but it must be findable.

If, for instance, $f(t)$ can be expanded as a convergent power series

$$f(t)=\sum_{r=a}^{w} C_r\frac{t^r}{r!}+R_w(t),$$

where $R_w\to 0$ as $w\to\infty$, it must follow that

$$pF(p)=\sum_{r=a}^{w} C_rp^{-r}+Z_w(p),$$

where $Z_w(p)H(t) \to 0$ as $w \to \infty$. Other cases will need further consideration.

Borel's theorem can be established on the same lines, since, if we have two identities in p,

$$F_2(p) = \int_0^\infty f_2(x) e^{-px}\, dx, \quad \text{and} \quad F_3(p) = \int_0^\infty f_3(y) e^{-py}\, dy, \quad . \quad . \quad . \quad . \quad 12_a$$

we can proceed

$$pF_2(p)F_3(p)H(t) = \int_0^\infty f_2(x)\, dx \int_0^\infty f_3(y)\, dy\, p \,.\, e^{-p(x+y)} H(t)$$

$$= \int_0^\infty f_2(x)\, dx \int_0^\infty f_3(y) \frac{dH(t-x-y)}{dt} dy, \quad . \quad . \quad . \quad . \quad 12$$

and since x and y are each positive, the integral must vanish either if $y > t - x$ or if $x > t$. Thus the upper limit of x may be put t, and Fourier's theorem gives us

$$pF_2(p)F_3(p)H(t) = \int_0^t f_2(x) f_3(t-x)\, dx, \quad . \quad . \quad 12_c$$

so that if

$$pF_s(p)H(t) = f_s(t) \quad \text{for} \quad s = 1, 2, \text{ or } 3,$$

and if

$$F_1(p) = F_2(p)F_3(p),$$

we have

$$f_1(t) = \int_0^t f_2(x) f_3(t-x)\, dx$$

$$= F_2(p) f_2(t) = F_3(p) f_2(t) \,. \quad . \quad . \quad 12_d$$

Thus, from three integral equations we can deduce a known theorem having no apparent connexion with them, by using the operator axiom in conjunction with the $H(t)$ function and Fourier's theorem. This does not prove the axiom, but it indicates the possibilities of its justification. Borel's theorem seems to show that $F_1(p)$ can be manipulated in any way consistent with its remaining algebraically identical with the product $F_2(p)F_3(p)$ without altering $f(t)$, provided that the functions are continuous throughout the range used for the variable. The use of $H(t)$ compels this range to be positive.

Carson bases his operational calculus* on two fundamental formulæ: (i.) the integral equation, and (ii.) a formula

* 'Electric Circuit Theory,' *loc. cit.* pp. 16, 21, 23.

involving $A(t)$—a quantity based upon the $H(t)$ function whose properties are assumed.

If the latter function be formulated and used explicitly Carson's second formula will be found to be simply a case of Borel's theorem.

$A(t)$ is the current at time t caused by a "unit" electromotive force applied at and from $t=0$; thus,

$$A(t)=\Delta(p)H(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad 13_a$$

where $\Delta(p)$ is some function of the operator p. It follows that

$$A(t-\tau)=\Delta(p)H(t-\tau),$$

and this is non-existent till t equals or exceeds τ.

If the electromotive force at and from $t=0$ is $E(t)$, we must similarly have

$$E(t)=\phi(p)H(t) \quad . \quad . \quad . \quad . \quad . \quad . \quad 13_b$$

The current $I(t)$ should be obtainable from (13 a) by substituting $E(t)$ for $H(t)$, giving

$$I(t)=\Delta(p)E(t)=\Delta(p)\phi(p)H(t)=\phi(p)A(t), \quad . \quad 13_c$$

and this is the precise value given by Carson's analysis, based on the idea that $I(t)$ is the result of increasing $E(t)$ by infinitesimal increments dE at $t=\tau$, leading to the equation

$$I(t)=E(0)A(t)+\int_0^t A(t-\tau)\frac{dE}{d\tau}d\tau,$$

which can readily be transformed to Carson's second formula

$$I(t)=\frac{d}{dt}\int_0^t E(\tau)A(t-\tau)\,d\tau \quad . \quad . \quad . \quad . \quad 13_d$$

Now 13_c states

$$I(t)=\frac{d}{dt}\left[p\frac{\Delta(p)}{p}\frac{\phi(p)}{p}H(t)\right],$$

and by using the preceding theorem with

$$\Delta(p)=pF_2(p), \quad \phi(p)=pF_3(p),$$

we get 13_d.

When the operand is an index function the operator axiom is rarely needed, even in cases when we have closed expressions in p involving binomial or exponential functions of p. Certain conditions must be fulfilled, the chief of which appears to be that the operator must not include a term having as a factor the reciprocal of a binomial expression

in p. Such an inverse operator must be dealt with by means of Boole's partial fraction solution.

If Δ_1 and Δ_2 are any polynomials in p, and if Δ^n means the operator Δ used n times in succession, it is easy to show

$$\Delta_1{}^r\Delta_2{}^s=\Delta_2{}^s\Delta_1{}^r, \quad . \quad . \quad . \quad . \quad . \quad . \quad 14_a$$

$$\operatorname{Exp}(a\Delta_1) . \operatorname{Exp}(b\Delta_2)=\operatorname{Exp}(a\Delta_1+b\Delta_2), \quad . \quad . \quad 14_b$$

$$\Delta^r=(p+a)^{mr} \quad \text{if} \quad \Delta=(p+a)^m, \quad . \quad . \quad . \quad 14_c$$

provided r, s, m are all positive integers.

If, however, m is fractional or negative, the meaning of Δ has to be defined, and in such a way as to distinguish between the two expansions

$$(p+a)^m=p^m+m\,p^{m-1}a+\text{ etc.},$$
$$(a+p)^m=a^m+m\,a^{m-1}p+\text{ etc.},$$

since the expansion, as an infinite series of power terms in p, has all the indices of p, in one case fractional and in the other case integral, whenever m is fractional.

Bearing this distinction in mind, by multiplying out the two series, and by collecting coefficients of like powers of p, it is possible to prove, in the same way as if β and γ were positive integers, that

$$\left.\begin{aligned}&(p+a)^\beta(p+a)^\gamma=(p+a)^{\beta+\gamma},\\ \text{and}\qquad &(a+p)^\beta(a+p)^\gamma=(a+p)^{\beta+\gamma},\end{aligned}\right\} \quad . \quad . \quad . \quad 14_d$$

whatever β or γ, provided only $\beta+\gamma>-1$; that is, provided the resultant operator does not contain a factor which is an inverse binomial. This can be done by using the summation formula for the hypergeometric series, when this formula is thrown into the form of the addition theorem * for binomial coefficients

$$\frac{(\beta+\gamma)!}{(\beta+\gamma-\alpha)!}=\sum_{n=0}^{\infty}\frac{1}{n!}\,\frac{\alpha!}{(\alpha-n)!}\,\frac{\beta!}{(\beta-n)!}\,\frac{\gamma!}{(\gamma+n-\alpha)!}, \quad . \quad 15_a$$

or

$$\binom{\beta+\gamma}{\alpha}=\sum_{n=0}^{\infty}\binom{\beta}{n}\binom{\gamma}{\alpha-n}, \quad . \quad . \quad . \quad . \quad . \quad 15_b$$

with the condition for convergency

$$\beta+\gamma>-1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 15_c$$

* In "Heaviside's Fractional Differentiator," *loc. cit.* pp. 409–414, a discussion of this formula will be found, together with some applications to closed expressions involving p.

As regards multiplying out the two infinite series, divergency questions do not arise in the usual way, since each of the terms involving p is an operator and has no numerical value, either absolute or relative, while there is no such thing as a radius of convergence for p. The restraints of divergency appear, however, in another form, since, unless 15_c holds, the hypergeometric series 15_a, representing the coefficient of a given power of p, is divergent, so that the summation formula does not apply.

Subject to this condition 15_c binomial expressions involving p can be manipulated in the ordinary way as if p were a numerical quantity, and this is even the case if instead of p we use $(p^x+y)^z$, where x, y, and z are any numerical quantities (with z positive).

Similarly in the case of exponential expansions 14_b holds if Δ_1 and Δ_2, instead of being ordinary polynomials, are binomial expansions, provided that it is always certain that Δ^s, for any positive integer s, is a perfectly definite power series. Thus Δ_1 may be the expansion $(p^\alpha+a)^\beta$ where β is any positive quantity, since $\Delta_1{}^s$ by 14_d always means the same as $(p^\alpha+a)^{\beta s}$, so that $\exp \Delta_1$ becomes perfectly definite; but if β is negative the terms with sufficiently high values of s will correspond with powers less than -1, and will thus involve inverse polynomials needing special treatment as above stated. Subject to this condition, exponential expansions can be treated algebraically. Thus, one of Heaviside's operators

$$\operatorname{Exp}\left[-\frac{x}{v}(p^2-\sigma^2)^{\frac{1}{2}}\right]$$

can be so treated, since the power of $(p^2-\sigma^2)$ is positive.

From the foregoing it seems to follow that the justification of the operator axiom rests upon two points: (i.) the fundamental operator p must have all the characteristics of an algebraic quantity, and (ii.) the manipulation of a function of p must be carried out with strict attention to the requirements of identity. The first point rejects the differentiator of Boole in favour of the index operator of Heaviside, clears the subject from dependence on the calculus, and makes intelligible the use of fractional powers of p. The second point merely means that when expanding a function of p in the form of a series S_n the remainder term R_n must never be omitted. Divergency questions do not arise with the operator series S_n, but appear at once when this operator is applied to the operand G. The quantity series S_nG will be convergent with some operands and divergent with

others. In the former case R_nG will be negligible, and in the latter it will be of dominant importance. R_n must always be retained, since its function is to act as a safeguard against divergency.

Now it is a characteristic of contour integration that the remainder term never is neglected and that its effects are collected in the form of residues. It thus seems a fact that the fundamental contour integral

$$F(p) = \frac{1}{2\pi i}\int\frac{F(z)}{z-p}\,dz \quad . \quad . \quad . \quad . \quad . \quad 16_a$$

can be regarded as an identity in p, in the sense that it holds not only for any simple or complex number, but also for every algebraic quantity. This assumption seems to be taken for granted in some mathematical work, but, so far as I am aware, it has never been explicitly claimed as true. Bromwich appears to make this assumption when he interprets an operator $F(p)$ as

$$F(p) = \frac{1}{2\pi i}\int\frac{e^{zt}F(z)}{z}\,dz, \quad . \quad . \quad . \quad . \quad 16_b$$

with the real part of z always positive.

It will be found that p in 16_a can, whenever convenient, be treated either as a number or as an algebraic operator. Thus, if we differentiate successively with regard to p, and put $p=0$, we get the values needed for Maclaurin's expansion of $F(p)$ in ascending powers of p, which comes out at once on expanding $1/(z-p)$ as such a power series and on using the values previously obtained. Also, in 16_a, if the contour includes all the poles of $F(z)$, and if the residues are calculated after treating p as the operator, we get * the expansion theorem of Heaviside.

Again, if we treat $(16\,a)$ as an identity in p, and use each element as an operator with $H(t)$ as operand, we get

$$2\pi i F(p)H(t) = \int F(z)\,\frac{1}{p}\,\frac{p}{z-p}\,H(t)\,dz,$$

where

$$\frac{1}{p}\,\frac{p}{z-p}\,H(t) = -\frac{1}{p}\left(1+\frac{z}{p}+\ldots\right)H(t)$$

$$= -\frac{1}{p}e^{zt}H(t) = \frac{1-e^{zt}}{z}\,H(t),$$

and this yields 16_b with the constant term $F(0)$ cut out.

* See Jeffreys, Cambridge Tract No. 23, p. 19.

The assumed interpretation of the operator 16_b, and the identity assumption in reference to 16_a, thus seem interchangeable in the sense that if one assumption is admitted the other appears to follow.

If the identity assumption be granted, the justification of Heaviside's index calculus seems also to follow, since all he assumed was that an operator given in terms of p could be treated as if it were an algebraic function of p, and afterwards used in the new form as an equivalent operator in p.

As a final example of this we may instance the following case * taken from Heaviside's book, his argument being expanded by using $\mathrm{H}(t)$ explicitly.

Assume that an ordinary exponential integral

$$\int_0^\infty e^{-\lambda x}\,dx=\frac{1}{\lambda}$$

can, with λ positive, be treated as an identity in λ. Put $\lambda=1/p_1+1/p_2$. The identity must still hold for either p_1 or p_2. Take first $p_2=\infty$ and $p_1=d/dt_1$ with $\mathrm{H}(t_1)$ as operand. We get the impulse function

$$p_1\mathrm{H}(t_1)=\int_0^\infty e^{-\frac{x}{p_1}}\,\mathrm{H}(t_1)\,dx=\mathrm{H}(t_1)\int_0^\infty \mathrm{J}_0 2\sqrt{xt_1}\,dx$$

on expanding the exponential operator and using

$$\frac{1}{p_1^n}\mathrm{H}(t_1)=\mathrm{H}(t_1)\frac{t_1^n}{n!},\text{ etc.}$$

Next take $p_2=d/dt_2$, and use as operand $\mathrm{H}(t_1)\times\mathrm{H}(t_2)$. We have

$$\frac{p_1p_2}{p_1+p_2}\mathrm{H}(t_1)\mathrm{H}(t_2)=\int_0^\infty e^{-x\left(\frac{1}{p_1}+\frac{1}{p_2}\right)}\mathrm{H}(t_1)\mathrm{H}(t_2)\,dx.$$

The first element becomes

$$\begin{aligned}p_2\left(1-\frac{p_2}{p_1}+\frac{p_2^2}{p_1^2},\text{ etc.}\right)\mathrm{H}(t_1)\mathrm{H}(t_2)&=\mathrm{H}(t_1)p_2e^{-t_1p_2}\mathrm{H}(t_2)\\&=\mathrm{H}(t_1)\frac{d}{dt_2}\mathrm{H}(t_2-t_1),\end{aligned}$$

while the second element becomes

$$\begin{aligned}&\int_0^\infty e^{-\frac{x}{p_2}}\mathrm{H}(t_2)\mathrm{H}(t_1)\mathrm{J}_0 2\sqrt{xt_1}\,dx\\&\qquad=\mathrm{H}(t_1)\int_0^\infty \mathrm{H}(t_2)\mathrm{J}_0(2\sqrt{xt_2})\mathrm{J}_0(2\sqrt{xt_1})\,dx.\end{aligned}$$

* See 'Electromagnetic Theory,' iii. p. 238. Many other (and more complex) cases will be found in the following fifty pages.

Now by Fourier's theorem

$$f(t_2)=\int_0^\infty f(t_1)\frac{d}{dt_2}\mathrm{H}(t_2-t_1)\,dt_1,$$

and with the aid of the above relations this becomes

$$f(t_2)=\int_0^\infty f(t_1)\,dt_1\int_0^\infty \mathrm{J}_0(2\sqrt{xt_1})\mathrm{J}_0(2\sqrt{xt_2})\,dx,$$

provided t_1, t_2, and x are each positive. Heaviside establishes (pp. 238, 239) a similar theorem for the J_m functions, and shows, as a variant of the interchange form of Fourier's Theorem, that

if

$$\mathrm{F}(y)=\int_0^\infty f(x)\mathrm{J}_m(2\sqrt{xy})\,dx,$$

then

$$f(x)=\int_0^\infty \mathrm{F}(y)\mathrm{J}_m(2\sqrt{yx})\,dy.$$

XVII. *Ionization in Gas-filled Photoelectric Cells :* (i.) *The Inert Gases in Cæsium on Silver Photoelectric Cells, and* (ii.) *Time Lag in Gas-filled Photoelectric Cells. By* W. F. Tedham, *M.A.**

(i.) The Inert Gases in Cæsium on Silver Photoelectric Cells.

Introductory.

SOME uncertainty appears to exist as to the best gas-filling in use in photoelectric cells which are sensitive to visible or infra-red light. After a comparison of helium, neon, and argon as fillings in alkali metal photoelectric cells, Kunz and Stebbings (1) concluded that "the best results were obtained with rubidium and neon." L. R. Koller (2) observes : "Argon is most satisfactory due to its low ionizing potential, but neon and helium are also used."

Campbell (3) says, "the usual filling gases are argon and the mixture of neon and helium from the air. The pressure is largely fixed by the glow potential required."

Other work on this problem includes that of E. F. Seiler (4), who noted that the maximum amplifying effects for different gases were obtained at different pressures. L. R. Koller

* Communicated by Dr. W. H. Eccles, F.R.S.

and H. A. Breeding [5] have shown curves for argon-filled cells with cæsium cathodes, while V. Zworykin [6] gives the colour-sensitivity curves for cæsium on magnesium cathodes when filled with the gases helium, neon, and argon at noted pressures.

In view of this uncertainty, the question of the relative advantages of helium, neon, and argon, and also krypton and xenon as gas-fillings, in photoelectric cells was reinvestigated in some experiments made in April 1930.

Gas Theory.

The amplifying effect of a gas depends on at least six parameters :—

V, the potential between the electrodes ;

I, the ionization potential of the gas ;

η, the ionizing efficiency of the electrons (at velocities 0 to v) ;

α, the coefficient of recombination of the ions ;

u, the mobility of the ions ; and

p, the pressure.

In any given cell the size and geometry of the electrodes will also determine the gas current. The ionizing efficiency η, which is a function of electron velocity, is a parameter concerning which very little is known. In any gas α and η are related, and α and u are both dependent on the pressure. Gas theory, therefore, indicates that the amplifying effect of a given gas in a photoelectric cell depends on V the potential between the electrodes and on p the pressure—a conclusion which is in agreement with experiment.

Hence the total current i through photoelectric cells of the same shape depends on six variables :—

ϵ (microamps. per lumen), the emissivity of the cathode;

λ (angströms), the incident wave-length ;

I (lumens), the quantity of light incident ;

V (volts), the potential between the electrodes ;

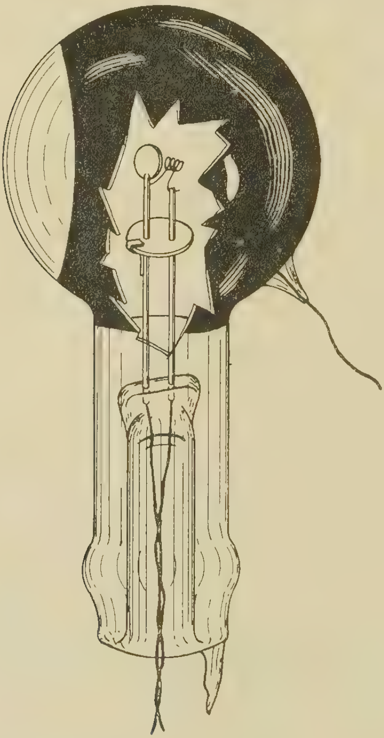
r (cms.), the distance between the electrodes ; and

p (microns mercury), the gas pressure.

The method adopted in a series of experiments with the five inert gases from the air was to take current voltage

curves for a measured pressure, reduce the pressure by pumping out gas, and take a fresh series of points. With ϵ, λ, I, and r held fixed, it is impossible to generalize too widely on the properties of these gases in photoelectric cells. Enough is known, however, to affirm that the results of these experiments on gas-filling apply in essentials to cells

Fig. 1.



Photocell. (About $\frac{2}{3}$ size.)

of different size, shape, and cathode, and with different illumination.

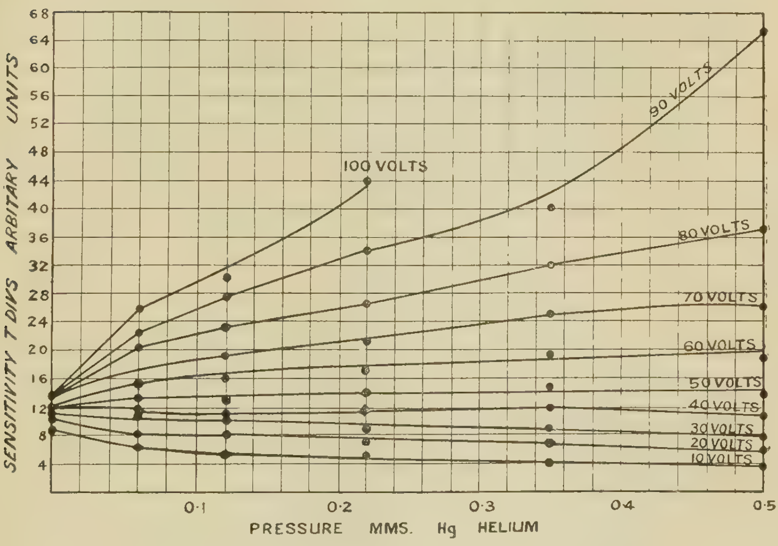
Experimental.

The cell used was spherical, 6 cm. in diameter (fig. 1), and had a cæsium on silver cathode. It was of the early type, in which the colour-sensitivity curve has a peak towards the blue end of the spectrum. It was illuminated through a

circular window 1 inch in diameter from a small gas-filled lamp carrying constant current. The illumination received by the cell was 0·16 lumen. The photoelectric cell-currents were measured on a Tinsley galvanometer, sensitivity approximately eleven divisions per microampere. The voltages were applied to the cell by means of a potentiometer with a series resistance of 30,000 ohms to diminish damage to the cell in the event of a glow discharge.

Purification of the helium, neon, and argon used was effected by heating them in turn over a previously evacuated mixture of potassium, magnesium, and calcium oxide in a

Fig. 2.



Comparison of inert gases in photocells—Helium.

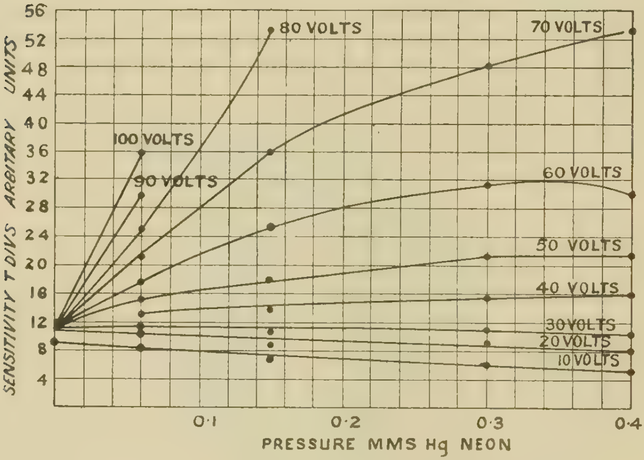
quartz tube. These three gases, as well as the krypton and xenon used, were supplied by the British Oxygen Company. The vacuum work was done on a pump assembly comprising a three-stage mercury vapour pump backed by a Hyvac pump. Pressures were measured on a pair of McLeod gauges.

Experimental Results.

The results (figs. 2–6) show several interesting features. All the gases depress the vacuum sensitivity when the anode potential is 20 volts or less. At 30 volts helium and neon

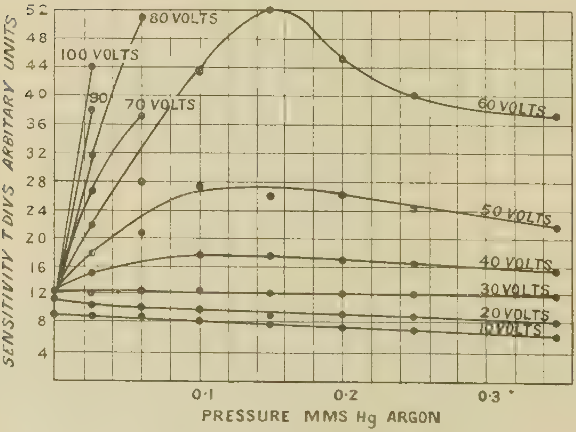
depress the sensitivity, while argon scarcely affects it. Krypton first increases the sensitivity with a maximum at 80 (microns of mercury) pressure and then depresses

Fig. 3.



Comparison of inert gases in photocells—Neon.

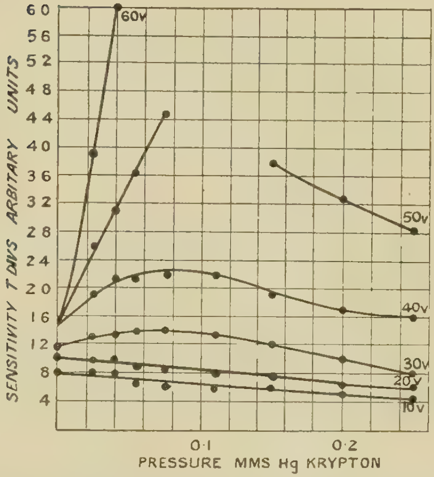
Fig. 4.



Comparison of inert gases in photocells—Argon.

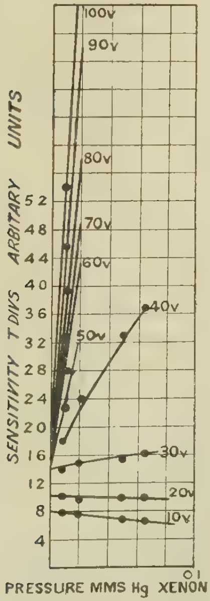
it at higher pressures. Xenon at 30 volts shows amplification only at the pressure used. At higher voltages all the gases show amplification effects and many of the curves show a maximum.

Fig. 5.



Comparison of inert gases in photocells—Krypton.

Fig. 6.



Comparison of inert gases in photocells—Xenon.

Helium.—At 80 volts helium shows maximum amplification at 0·5 mm. pressure.

Neon.—At 80 volts neon could not be taken over the whole range of pressures, as gas-glow sets in ; but at 70 volts the maximum amplification occurred at 0·3 mm. pressure.

Argon.—At 70 volts argon could not be taken over the whole range of pressure ; at 60 volts maximum amplification occurred at 0·15 mm. pressure.

Krypton.—At 60 volts krypton could not be taken over the whole range ; at rather less than 50 volts the maximum amplification occurs at 0·1 mm. pressure.

Xenon.—The curves for xenon were limited by this gas condensing in the liquid-air trap, and hence do not show exactly where the maxima occur. The 40-volt curve for this gas is almost coincident with the 50-volt curve for krypton.

The figures show that, for the inert gases in these cells, an amplification of five is usually practical at illuminations as high as 0·16 lumen and with a cathode sensitivity of about ·6 microamperes per lumen. By using low pressures and high voltages greater amplification is obtained for a gas of higher atomic number.

I have seen no explanation of the mechanism by which gas-filling diminishes the cell current at the lower voltages. The process may simply be due to the target formed by the gas itself.

(ii.) Time Lag in Gas-filled Photoelectric Cells.

It is well known that the alternating current response of gas-filled photoelectric cells to light of periodically varying intensity diminishes with the frequency. This decrease may become apparent at frequencies from 700 p.p.s. upward.

I first noticed the effect some time ago with two potassium hydride neon-filled cells supplied by Messrs. Otto Pressler of Leipzig. These cells, of the banjo type, were similar geometrically, but had slightly different current-voltage curves indicating differences in gas pressure. When calibrated in the same amplifier by a standard alternating light source [(7)], the frequency output curves were markedly different (figs. 7 & 8). These two curves show there is a phenomenon additional to the drooping response characteristic, due to the diminishing input impedance, of most

amplifiers at high frequencies [8]. One of the early descriptions of time lag is due to P. Toulon [9].

It is also known [10], [11] that the interval between the incidence of light and the emission of the equivalent number of electrons is of the order 3×10^{-9} sec., and calculation of the time occupied by the gas ions reaching the electrodes [12] gives a value about 10^{-5} sec. This time does not readily account for a variation with frequency occurring below 10^3 cycles sec.

When an interrupted light-beam falls on a gas-filled cell the total charge passing through the cell per unit time is always proportional to the total light-incident, whatever the frequency. In other words, in a chopped light-beam the

Fig. 7.

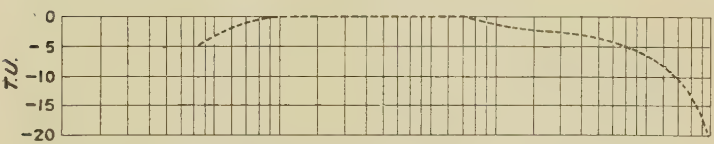


Diagram showing relative response curve of amplifier 864. Optical input Pressler cell 2809/801.

Fig. 8.

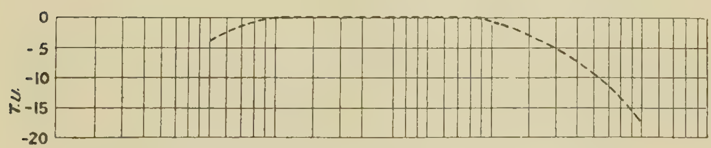


Diagram showing relative response curve of amplifier 864. Optical input Pressler cell 2809/804.

mechanism which delays the rise of current in a gas-filled cell is capable of liberating the missing number of ions after the illumination is cut off.

The source of the stored ionizing energy cannot easily be charged particles, since these would be collected by the field. It is unlikely to be the cathode, since the effect is only present when gas is present and increases with the pressure and the electric field. The effect occurs in cells with all known cathodes.

It is necessary, then, to look for some effect with a life of 10^{-4} to 10^{-3} sec. in the inert gases which occurs in the presence of an electron stream. The useful summary of

gas-discharge work [13] enumerates the ionizing processes in gases as :—

(1) Ionization by electron impact.

(2) Photo-ionization.

(3) Cumulative ionization.

(4) Positive ion impact.

(5) Collisions of the second kind.

(6) Thermal ionization.

Processes (2), (4), (5), and (6) do not occur in the gas of photoelectric cells as ordinarily used. Process (1) is usually accredited with all the amplification produced in gas-filled cells. It is suggested that process (3), which includes the formation of metastable atoms, may account for the time lag. Dorgelo [14] has observed metastable atoms in neon with lives of 1/240 and 1/2000 sec. This life is of the order required to explain the time lag.

As the pressure in gas-filled cells nowadays varies from about 4–25 microns of mercury, optical observation of metastable atoms would be very difficult. I should be grateful if anyone would suggest a simple method of testing this hypothesis.

My acknowledgments are due to The Gramophone Company in whose laboratories the work on gas-filling was done, and especially to Mr. H. Neal, who carried out the experimental work with patience and skill.

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XVIII. *On some Problems in the Conduction of Heat.* By GEORGE GREEN, *D.Sc., Lecturer in Physics in the Applied Physics Department of the University of Glasgow* *.

IN a series of papers † under the above title examples are given of an entirely general method of investigating the solutions of all problems involving the transmission of an effect by wave-motion throughout a limited medium. The special feature of the method lies in the procedure adopted for determining the effect of boundary conditions. In all branches of physics involving wave-motion it is always possible to find a solution to represent a periodic disturbance throughout an infinitely extended medium emanating from a definite source, such as a plane or cylindrical or spherical surface, a line, or a point, according to the nature of the problem to be solved. Summations of such solutions can then be obtained to represent initial conditions and conditions varying with time. In the application of the present method each of these fundamental solutions represents a wave-train which is to be regarded as continuing by transmission or by reflexion at the boundaries specified in any problem. The aim in view is therefore to follow completely each particular train from its source, to find its effect at a given point before and after reflexion at a given boundary, or after successive reflexions at various boundaries, and so arrive at its complete effect at the given point by summation. Corresponding to each particular type of source employed, the plane, the line, or the point, two wave-trains can always be obtained representing effects diverging from the source and two likewise representing effects converging to the source. As plane or line sources can themselves be built up from point sources, it is clear that for each medium there are four fundamental wave-trains, and the solution of all problems involving transmission of effects by wave-motion within the medium can be expressed in terms of these four fundamental wave-trains. The same statement holds in the case of physical properties or conditions within a medium which are not of themselves vibratory, but are represented mathematically by summations of Fourier wave-trains which can be regarded as representing a wave-motion which may or may not be of some ultimate significance in relation to the properties or conditions with which they are concerned.

* Communicated by the Author.

† Phil. Mag. iii. Suppl. (April 1927); v. (April 1928); ix. (February 1930).

The method employed throughout these papers is therefore an entirely general one, and there are no doubt interesting problems in the theory of sound and other subjects yet to be examined analogous with the problems in heat conduction now under discussion. The fundamental importance of solutions relating to a point source is evident, and the intention of the present paper is to extend the usefulness of the method employed by the discussion of several problems involving wave-motion to or from a point source. In general the solutions arrived at are in the form of integrals. All the well-known heat solutions, for example, can readily be expressed in integral form by the present method.

For the purpose in view it is better to obtain the solutions we require direct from the differential equation applicable to the conduction of heat in a uniform medium. For the case of symmetry about a given axis this equation takes the form

$$\frac{\partial v}{\partial t}=\kappa\left\{\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)\right\}, \quad (1)$$

where θ is the angle between the axis of symmetry and the radius vector r drawn from the origin O to any point P, and κ represents $K/c\rho$ in a well-known notation. The solutions required are those for which temperature, v, varies with time according to the factor e^{ikt}, where $k/2\pi$ is the frequency of a regular periodic vibration. To determine v as a function of r and θ we have then the equation

$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}-\frac{ik}{\kappa}v+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right)=0. \quad (2)$$

If we now, following the usual procedure, put $v=(R/r^{\frac{1}{2}})S$, where R represents a function of r only and S represents a function of θ only, we find that, when S denotes $P_m(\cos\theta)$, the zonal harmonic of order m, the equation to determine R becomes

$$\frac{\partial^2 R}{\partial r^2}+\frac{1}{r}\frac{\partial R}{\partial r}-\left(\frac{ik}{\kappa}+\frac{(m+\frac{1}{2})^2}{r^2}\right)R=0. \quad . \quad . \quad (3)$$

As the general solution of this equation is given by

$$R=A\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)+B\,I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right), \quad . \quad (4)$$

where A and B represent arbitrary constants, we accordingly obtain as general type solutions for the equation above the expressions

$$v=Ae^{ikt}\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)P_m(\cos\theta)/r^{\frac{1}{2}},$$

and $$v = \mathrm{B}e^{ikt}\,\mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\mathrm{P}_m(\cos\theta)/r^{\frac{1}{2}}. \quad . \quad . \quad (5)$$

One of these functions, $\mathrm{K}_{m+\frac{1}{2}}(z)$, is infinite in value when $z=0$, and tends to the value zero when z is indefinitely increased. The other function, $\mathrm{I}_{m+\frac{1}{2}}(z)$, has the value zero when $z=0$, and becomes infinite in value as z increases indefinitely. The real and imaginary parts of the first expression given in (5) represent two wave-trains diverging from the origin, and the real and imaginary parts of the second expression given in (5) represent two negative wave-trains converging to the origin.

In the case of complete symmetry about the origin $m=0$,

$$\mathrm{K}_{\frac{1}{2}}(z) = \frac{\pi}{\sqrt{2\pi z}}e^{-z} \quad \text{and} \quad \mathrm{I}_{\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}(e^{z}-e^{-z}). \quad (6)$$

If, for example, a source of heat exists at the origin at which there is a periodic heat supply qe^{ikt}, the solution of (1) required to represent this involves only the diverging wave-trains, and must therefore be of the form

$$v = \mathrm{A}e^{ikt}\,\mathrm{K}_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/ r^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (7)$$

We can determine A from the condition that

$$\underset{r\to 0}{\mathrm{Lt}}\left(-4\pi r^2\mathrm{K}\frac{\partial v}{\partial r}\right) = qe^{ikt}, \quad . \quad . \quad . \quad . \quad (8)$$

and we then find that the temperature due to a periodic source of strength qe^{ikt} situated at the origin is represented by

$$v = \frac{q}{4\pi\mathrm{K}}\sqrt{\frac{2}{\pi}\frac{ik}{\kappa}}\,e^{ikt}\,\mathrm{K}_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}\,r\right)^{\frac{1}{2}}. \quad (9)$$

Suppose next that a periodic heat supply of amount qe^{ikt} per unit area occurs at a spherical surface of radius $r=r_1$, then two expressions are required to represent the temperature conditions outside and inside the surface $r=r_1$ respectively. The solution required involves both types of wave-trains, namely

$$v_0 = \mathrm{A}e^{ikt}\,\mathrm{K}_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/ r^{\frac{1}{2}} \quad . \quad . \quad r>r_1, \quad . \quad (10)$$

$$v_i = \mathrm{B}e^{ikt}\,\mathrm{I}_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/ r^{\frac{1}{2}} \quad . \quad . \quad . \quad r_1>r. \quad . \quad (10')$$

It is worthy of remark here that the negative wave-train

emitted by the source passes through the origin and thereafter becomes a positive or diverging wave-train. That v_i contains both the negative and the positive continuation is evident from the form of $I_{\frac{1}{2}}(z)$ given in (6) above. The constants A and B are readily determined from the conditions

$$v_i = v_0 \quad \text{and} \quad -K\left(\frac{\partial v_0}{\partial r} - \frac{\partial v_i}{\partial r}\right) = qe^{ikt} \quad \text{at} \quad r = r_1. \quad (11)$$

Accordingly we find that the solution representing a periodic surface source of strength qe^{ikt} per unit area situated at $r=r_1$ is given by

$$v_0 = \frac{qr_1}{K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}} e^{ikt} K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right) I_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r_1\right). \quad . \quad r > r_1, \quad (12)$$

$$v_i = \frac{qr_1}{K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}} e^{ikt} K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r_1\right) I_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right). \quad . \quad r_1 > r. \quad (12')$$

When r and r_1 become indefinitely great, with $r-r_1$ finite and equal to x, the upper expression represents the solution corresponding to a plane periodic surface source of strength qe^{ikt} per unit area, situated at $x=0$.

The solutions given above are similar to solutions obtained in a previous paper (Phil. Mag. v. pp. 700–720 (April 1928)), where corresponding problems relating to the line source and to cylindrical surface sources are discussed. Equations (9) and (12) above correspond with (8) and (16) respectively of the paper referred to. The similarity in the solutions obtained for corresponding problems involving point and line sources respectively is maintained when we pass in each case from a periodic source to the corresponding instantaneous source, and in fact throughout. For an instantaneous heat supply q at the origin, at $t=0$, the temperature at any point r at time t is represented by

$$v = \frac{q}{4\pi^2 K}\sqrt{\frac{2}{\pi}}\int_0^\infty dk\, e^{ikt}\sqrt{\frac{ik}{\kappa}}\, K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}\,r\right)^{\frac{1}{2}} \quad . \quad . \quad (13)$$

$$= \frac{q\kappa e^{-\frac{r^2}{4\kappa t}}}{8K(\pi\kappa t)^{\frac{3}{2}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13')$$

by an evaluation given in equations (4) and (13) of the Phil. Mag. paper, iii. Suppl. (April 1927). Similarly, if we take the integral $\frac{1}{\pi}\int_0^\infty dk\, v$, where v is given by (12) above, we

find that the temperature, due to an instantaneous heat source of amount q per unit area at the surface $r=r_1$, is given by

$$v=\frac{qr_1}{\pi K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}}\int_0^\infty dk\, e^{ikt}\,K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)I_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r_1\right). \quad . \quad r>r_1, \quad . \quad . \quad . \quad (14)$$

or by the corresponding expression with r and r_1 interchanged under the integral sign when $r_1>r$.

Solutions similar in type to (12) and (14) are obtained when we consider sources involving zonal harmonics to express a variation in the strength of the source according to its position relative to an axis of symmetry on the surface $r=r_1$. Thus the solution representing a periodic surface source of strength $qe^{ikt}P_m(\cos\theta)$ per unit area situated at the surface $r=r_1$ is given by

$$v_0=\frac{qr_1}{K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}}e^{ikt}\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r_1\right)P_m(\cos\theta), \quad . \quad . \quad . \quad (15)$$

$$v_i=\frac{qr_1}{K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}}e^{ikt}\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r_1\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)P_m(\cos\theta), \quad . \quad . \quad . \quad (15')$$

and the corresponding solution for an instantaneous heat source of strength $q\,P_m(\cos\theta)$ per unit area at the same surface is given by

$$v=\frac{qr_1}{\pi K}\left(\frac{r_1}{r}\right)^{\frac{1}{2}}P_m(\cos\theta)$$

$$\int_0^\infty dk\, e^{ikt}\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r_1\right). \quad . \quad r>r_1, \quad (16)$$

or by the expression obtained by interchanging r and r_1 under the integral sign when $r_1>r$.

Two methods are available for the evaluation of the group of integrals of which (14) and (16) contain particular cases. In one method we may follow the procedure adopted in the paper already referred to to evaluate similar integrals appearing in similar problems relating to the cylinder. From (13′) we may find the solution corresponding to an instantaneous circular line source of strength q per unit length. In this replace q by $qr_1\sin\theta\, d\theta\, P_m(\cos\theta)$, and integrate over the complete spherical surface $r=r_1$. The

result of this integration is then the equivalent solution to that given in (16) above. The alternative method is more convenient for our present purpose. In it we start from the evaluation obtained in the case of a point source situated at the origin given in equation (13′) above. Suppose the point source referred to to be situated at a point O′ at distance r_1 from the origin O along the axis of symmetry; then, P being any point, we have $OP=r$ and $O'P=r'$, where $r'^2=r^2+r_1^2-2rr_1\cos\theta$. Equation (13′) takes the form

$$\int_0^\infty dk\, e^{ikt}\sqrt{\frac{ik}{\kappa}}\,K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r'\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}\,r'\right)$$

$$=\frac{\pi e^{-\frac{r^2+r_1^2}{4\kappa t}}}{2\sqrt{2}\kappa^{\frac{1}{2}}t^{\frac{3}{2}}}\,e^{\frac{rr_1}{2\kappa t}\cos\theta} \quad . \quad . \quad (17)$$

Now make use of the addition theorem for Bessel functions, according to which

$$\frac{K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r'\right)}{\left(\sqrt{\frac{ik}{\kappa}}\,r'\right)^{\frac{1}{2}}}=\left(\frac{2\pi}{\frac{ik}{\kappa}rr_1}\right)^{\frac{1}{2}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)$$

$$I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r_1\right)P_m(\cos\theta). \quad r>r_1, \quad . \quad (18)$$

and expand the expression on the right-hand side of (17) by means of the series

$$e^{\frac{rr_1}{2\kappa t}\cos\theta}=\left(\frac{\pi\kappa t}{rr_1}\right)^{\frac{1}{2}}\sum_{m=0}^{m=\infty}(2m+1)\,I_{m+\frac{1}{2}}\left(\frac{rr_1}{2\kappa t}\right)P_m(\cos\theta). \quad . \quad (19)$$

When corresponding terms on the two sides of the equation are equated it will be found that

$$\int_0^\infty dk\, e^{ikt}\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r_1\right)$$

$$=\frac{\pi}{2t}e^{-\frac{r^2+r_1^2}{4\kappa t}}\,I_{m+\frac{1}{2}}\left(\frac{rr_1}{2\kappa t}\right). \quad . \quad \begin{pmatrix}r>r_1\\ r_1>r\end{pmatrix}. \quad (20)$$

That a like result holds when m is substituted for $(m+\frac{1}{2})$ throughout (20) may be proved by repeating the above process with all expressions referring to a point source replaced by the corresponding expressions referring to a line

source *. By differentiating both sides of equation (20) with respect to r_1 or r additional evaluations are obtained.

Returning to (14), we can now write this solution in the form

$$v=\frac{qr_1}{2\mathrm{K}t}\left(\frac{r_1}{r}\right)^{\frac{1}{2}}e^{-\frac{r^2+r_1{}^2}{4\kappa t}}\mathrm{I}_{\frac{1}{2}}\left(\frac{rr_1}{2\kappa t}\right). \quad . \quad (21)$$

The well-known form of the solution for a spherical surface source of strength q per unit area at $r=r_1$ is recovered when the Bessel function is replaced by its equivalent given in (6) above. Equations (9) and (13′) for the point source solutions are likewise seen to be in agreement with well-known results. Further, if in (9) we replace q by $q\,dz$ and r^2 by (ρ^2+z^2), and integrate with respect to z from $-\infty$ to $+\infty$, we then arrive at the solution for a line source of strength qe^{ikt} units of heat per unit of length. Thus we obtain

$$v=\frac{q}{4\pi\mathrm{K}}\sqrt{\frac{2}{\pi}\frac{ik}{\kappa}}e^{ikt}\int_{-\infty}^{+\infty}dz$$

$$\mathrm{K}_{\frac{1}{2}}\left\{\sqrt{\frac{ik}{\kappa}(\rho^2+z^2)}\right\}\Big/\left(\sqrt{\frac{ik}{\kappa}(\rho^2+z^2)}\right)^{\frac{1}{2}}, \quad (22)$$

$$=\frac{q}{2\pi\mathrm{K}}e^{ikt}\,\mathrm{K}_0\left(\sqrt{\frac{ik}{\kappa}}\rho\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22')$$

by an evaluation given in Watson's 'Theory of Bessel Functions,' p. 417, thus confirming previous results.

The above solutions are concerned with various heat sources in an infinitely extended medium. We proceed now to consider the effects produced by various boundaries. Suppose, for example, that we wish to determine the temperature due to a periodic point source of strength qe^{ikt}, situated at a point O′ whose coordinates are $(r=a', \theta=0)$ within a sphere of radius a whose surface is maintained at zero temperature. The required solution obviously consists of two parts, one part, which we shall denote by v_1, representing the wave-system diverging from O′, and the other, which we shall denote by v_2, representing the wave-system continuing within the sphere after one or more reflexions from the spherical boundary $r=a$. The expression for v_1 is that given in (9), with r replaced by r', the radius from O′ to any point. But by means of the addition theorem already employed the wave-system diverging from O′ can be

* See Phil. Mag., April 1928, equations (37) and (38).

expressed as a system of waves diverging from O, the centre of the sphere. That is

$$v_1 = \frac{q}{2\pi K} \frac{e^{ikt}}{(ra')^{\frac{1}{2}}} \sum_{m=0}^{m=\infty} (m+\tfrac{1}{2}) K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right) I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) P_m(\cos\theta). \quad . \quad r > a', \quad (23)$$

and if in this we put $r=a$ we obtain the expression for the temperature existing at any point of the spherical boundary due to the point source alone. Now each positive constituent in the above system continues after one reflexion at the boundary as a negative train which, after passing through the origin, becomes a positive train which again returns to the boundary. Each train therefore builds up a wave-system within the sphere in a manner discussed in detail in a former paper *, the system consisting of positive and negative waves of the same type as in the original train. In this case the system built up by reflexions at the boundary must be of the type

$$e^{ikt}\left(\frac{a}{r}\right)^{\frac{1}{2}} \frac{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)}{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right)} P_m(\cos\theta), \quad . \quad . \quad (24)$$

and the boundary temperature due to each constituent of this type must be equal and opposite to the boundary temperature due to the corresponding constituent of v_1. Hence we find that the reflected wave-system is given by

$$v_2 = \frac{q}{2\pi K} \frac{e^{ikt}}{(aa')^{\frac{1}{2}}} \sum_{m=0}^{m=\infty} (m+\tfrac{1}{2}) K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right) I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) \left(\frac{a}{r}\right)^{\frac{1}{2}} \frac{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)}{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right)} P_m(\cos\theta), \quad . \quad (25)$$

and the complete solution of the problem is given by

$$v = v_1 - v_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

If the point source is situated outside a spherical boundary maintained at zero temperature ($a' > a$) the required solution

* Phil. Mag. (Feb. 1930).

can be derived from the above by interchanging $K_{m+\frac{1}{2}}$ and $I_{m+\frac{1}{2}}$ throughout v_1 and v_2. See equation (33) below.

The above process can evidently be employed whatever be the special boundary condition to be fulfilled at $r=a$. If the boundary condition be that

$$-K\frac{\partial v}{\partial r}=hv, \quad \text{at} \quad r=a, \quad . \quad . \quad . \quad (27)$$

the point source being within the sphere, all that is required is to add to the solution v_1 representing the diverging wave-system, the reflected wave-system consisting of negative or converging waves with their positive continuations, all of which are of the type indicated above in (24). Hence the required solution may be represented by

$$v=\frac{q}{2\pi K}\frac{e^{ikt}}{(ra')^{\frac{1}{2}}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\left[K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right)\right.$$

$$\left.+A_m I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\right]P_m(\cos\theta). \quad . \quad r>a, \quad (28)$$

where A_m represents a constant to be determined by the condition stated in (27). The value thus found for A_m is given by

$$A_m=-\frac{N_m}{D_m}I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

where

$$\left.\begin{aligned} N_m &= \sqrt{\frac{ik}{\kappa}}\,K'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)+\frac{h'}{K}K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right),\\ D_m &= \sqrt{\frac{ik}{\kappa}}\,I'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)+\frac{h'}{K}I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right),\\ &\text{and} \qquad h'=(h-K/2a). \end{aligned}\right\} \quad . \quad (29')$$

The numerator is the same function of $K_{m+\frac{1}{2}}$ that the denominator is of $I_{m+\frac{1}{2}}$.

An examination of the solutions given for the corresponding problems in which a periodic line source of strength qe^{ikt} per unit length is situated at a distance a' from the axis of an infinite cylinder of radius a reveals a very close similarity in form in the two sets of solutions. Compare (23), (25), (28), (29) above with equations (66), (67), (76), (77) respectively of the paper in Phil. Mag. v. (April 1928) already referred to. To pass from a solution applicable to a cylindrical boundary to the corresponding solution applicable to a spherical

boundary we replace $\cos m\theta$ by $(m+\frac{1}{2})\,P_m(\cos\theta)$, and each Bessel function of the type

$$K_m\left(\sqrt{\frac{ik}{\kappa}}\rho\right) \quad \text{by} \quad \left\{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)\Big/ r^{\frac{1}{2}}\right\}$$

in which the function is of the same type. These two modifications convert the solution applicable to a cylinder kept at zero temperature into the corresponding solution applicable to a sphere kept at zero temperature. In the case of the problems involving radiation at the boundary, in addition to the above modifications, the value of each boundary constant A_m applicable to the cylinder problem must be modified by inserting h' in place of h, to give the constant the value applicable to the corresponding problem for the sphere. The value of the comparison which we have made of the two sets of results for the cylinder and the sphere respectively lies in the fact that when we proceed to solutions of the type $\frac{1}{\pi}\int_0^\infty dk\, v$, where v represents any one of the solutions relating to a periodic point source already found, we can make use of the evaluations obtained for the corresponding integrals in the corresponding problems relating to line sources. These evaluations are given on pages 717–720 of the paper referred to. By integration of (26) with respect to k and the evaluation given in (75) of the previous paper we find that the temperature due to an instantaneous point source of strength q situated at a point at distance a' from the centre of a sphere of radius a whose boundary is maintained at zero temperature is given by

$$v = \frac{q\kappa}{\pi a^2 K}\frac{1}{(ra')^{\frac{1}{2}}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\,P_m(\cos\theta)$$
$$\left\{\sum_{s=1}^{s=\infty} e^{-\kappa\lambda_s^2 t}\frac{J_{m+\frac{1}{2}}(\lambda a')\,J_{m+\frac{1}{2}}(\lambda r)}{[J'_{m+\frac{1}{2}}(\lambda a)]^2}\right\}; \quad . \quad (30)$$

where λ denotes λ_s a root of the equation

$$J_{m+\frac{1}{2}}(\lambda a)=0.$$

Similarly, from (28) and (29) above and (85) of the previous paper, the solution for the corresponding problem in which radiation takes place at the boundary $r=a$ is given by

$$v = \frac{q\kappa}{\pi a^2 K}\frac{1}{(ra')^{\frac{1}{2}}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\,P_m(\cos\theta)$$
$$\left\{\sum_{s=1}^{s=\infty}\lambda^2 e^{-\kappa\lambda^2 t}\frac{J_{m+\frac{1}{2}}(\lambda a')\,J_{m+\frac{1}{2}}(\lambda r)}{\left(\frac{h'^2}{K^2}-\frac{(m+\frac{1}{2})^2}{a^2}+\lambda^2\right)[J_{m+\frac{1}{2}}(\lambda a)]^2}\right\}, \quad (31)$$

where λ denotes λ_s a root of the equation

$$K\lambda J'_{m+\frac{1}{2}}(\lambda a)+h'J_{m+\frac{1}{2}}(\lambda a)=0. \quad . \quad . \quad (31')$$

Problems similar to those dealt with above, but relating to a point source within a cylindrical boundary or in the neighbourhood of a plane boundary, can evidently be solved by the same method with the appropriate form of the addition theorem used for each boundary. In the case of a point source situated in the neighbourhood of a plane boundary the solutions can also be derived from solutions already obtained for a spherical boundary. When the plane is kept at zero temperature the solution required is evidently that in which an image source equal and opposite to the original source is placed behind the plane. It is instructive, however, to show that this result can be obtained from our former solution (26), relating to a spherical boundary. Consider the effect of two point sources, one of strength qe^{ikt} situated at the point O$'$, whose coordinates are $r=a'$, $\theta=0$, and the other, an equal and opposite source, situated at the point O$''$, whose coordinates are $r=a''$, $\theta=0$, where $a''>a>a'$. The solution consists of two sets of diverging waves represented by

$$v=\frac{q}{4\pi K}\sqrt{\frac{2}{\pi}\frac{ik}{\kappa}}\,e^{ikt}\left\{K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r'\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}r'\right)^{\frac{1}{2}}\right.$$
$$\left.-K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r''\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}r''\right)^{\frac{1}{2}}\right\}, \quad (32)$$

in which the first term is equivalent to the solution v_1 given in (23), (26), and (28) above, r' and r'' being radii from O$'$ and O$''$ respectively to any point P. The second of the above terms can in like manner be expanded by the addition theorem, which in this case, to represent the point source situated on the side of the spherical boundary ($r=a$) away from the origin, would give

$$v''=\frac{q}{2\pi K}\frac{e^{ikt}}{(ra'')^{\frac{1}{2}}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\,K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a''\right)$$
$$I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)P_m(\cos\theta) \quad . \quad . \quad a''>r, \quad (33)$$

and (32) can now be written in the form

$$v=v_1-v'', \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

for convenience in comparing it with the solution given in (26) above. Suppose now that the distances a', a, a'', r,

become indefinitely large, while $(a-a')$ and $(a''-a)$ remain finite and each equal to x_0. The two solutions represented by v_2 and v'' respectively are evidently equivalent provided

$$\frac{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a''\right)}{(a'')^{\frac{1}{2}}} = \frac{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a'\right)}{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a\right)(a')^{\frac{1}{2}}} \qquad (35)$$

Under the conditions stated we may replace each Bessel function by the first term of its asymptotic expansion, and it is easy to see that (35) holds good. Hence, in the limit when the boundary sphere becomes a plane, the reflected wave-system v_2 is equivalent to v'', the wave-system from a point source situated at the position of the image of the original point source in the plane.

Similarly, to determine the form of the wave-system reflected from a plane boundary, placed at distance x_0 from a point source of strength qe^{ikt}, under the condition that radiation takes place at the boundary to a medium at zero temperature, we have to investigate the form taken by the solution

$$v_2 = \frac{q}{2\pi K}\frac{e^{ikt}}{(ra')^{\frac{1}{2}}}\sum_{m=0}^{m=\infty}(m+\tfrac{1}{2})\,A_m\,I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)P_m(\cos\theta), \quad . \quad . \quad . \quad (36)$$

where A_m is given by (29), when a, a', and r become infinitely large with $(a-a')$ finite and equal to x_0. By making use of asymptotic values for the Bessel functions we may write

$$K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a\right) = \frac{\pi e^{-\sqrt{\frac{ik}{\kappa}}a}}{\sqrt{2\pi\left(\sqrt{\frac{ik}{\kappa}}a\right)}}$$

and

$$I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}a'\right) = \frac{e^{\sqrt{\frac{ik}{\kappa}}(a-x_0)}}{\sqrt{2\pi\left(\sqrt{\frac{ik}{\kappa}}(a-x_0)\right)}}$$

$$= \frac{e^{\sqrt{\frac{ik}{\kappa}}(a-x_0)}}{\sqrt{2\pi\left(\sqrt{\frac{ik}{\kappa}}a\right)}}. \quad . \quad (37)$$

We find accordingly that the value of A_m when a becomes infinite may be written in the form

$$A_m = \frac{\sqrt{\frac{ik}{\kappa}} - \frac{h}{K}}{\sqrt{\frac{ik}{\kappa}} + \frac{h}{K}} \frac{\pi \cdot e^{-\sqrt{\frac{ik}{\kappa}}(a+x_0)}}{\sqrt{2\pi\sqrt{\frac{ik}{\kappa}}(a-x_0)}} \quad . \quad . \quad . \quad (38)$$

$$= \left\{1 - \frac{2\frac{h}{K}}{\sqrt{\frac{ik}{\kappa}} + \frac{h}{K}}\right\} \frac{\pi e^{-\sqrt{\frac{ik}{\kappa}}(a+x_0)}}{\sqrt{2\pi\sqrt{\frac{ik}{\kappa}}(a-x_0)}}, \quad (38')$$

an expression which it is to be remarked does not depend on m, and is therefore the same in value for all the terms. It passes into the form we require if we make the substitutions

$$\frac{1}{\sqrt{\frac{ik}{\kappa}} + \frac{h}{K}} = \int_0^\infty e^{-\left(\sqrt{\frac{ik}{\kappa}} + \frac{h}{K}\right)\xi} d\xi\,;$$

$$\frac{\pi e^{-\sqrt{\frac{ik}{\kappa}}(a+x_0+\xi)}}{\sqrt{2\pi\sqrt{\frac{ik}{\kappa}}(a+x_0+\xi)}} = K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}(a+x_0+\xi)\right). \quad (39)$$

With these substitutions, and certain slight modifications depending on the condition that a is exceedingly great compared with any value of $x_0+\xi$ that need be considered, we find that we may take as an approximate expression for the reflected wave-system when the boundary sphere has exceedingly large radius

$$v_2 = \frac{qe^{ikt}}{2\pi K}\left[\frac{1}{\{r(a+x_0)\}^{\frac{1}{2}}} \sum_{m=0}^{m=\infty} (m+\tfrac{1}{2}) \right.$$
$$K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}(a+x_0)\right) I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)$$
$$-2\frac{h}{K}\int_0^\infty \frac{d\xi\, e^{-\frac{h}{K}\xi}}{\{r(a+x_0+\xi)\}^{\frac{1}{2}}} \sum_{m=0}^{m=\infty} (m+\tfrac{1}{2})$$
$$\left. K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}(a+x_0+\xi)\right) I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)\right] P_m(\cos\theta)\,.$$
$$. \quad . \quad . \quad (40)$$

A comparison of this expression with (33) above reveals that the terms contained in it represent point sources on the

opposite side of the plane boundary from the original point source, and we obtain finally as the solution, representing the temperature at any point due to a periodic point source of strength qe^{ikt}, situated at distance x_0 from an infinite plane across which radiation takes place to a medium at zero temperature, the expression

$$v = \frac{qe^{ikt}}{4\pi K}\sqrt{\frac{2}{\pi}\frac{ik}{\kappa}}\left\{\frac{K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)}{\left(\sqrt{\frac{ik}{\kappa}}r\right)^{\frac{1}{2}}} + \frac{K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r'\right)}{\left(\sqrt{\frac{ik}{\kappa}}r'\right)^{\frac{1}{2}}}\right.$$

$$\left.-\frac{2h}{K}\int_0^\infty d\xi\, e^{-\frac{h}{K}\xi}\frac{K_{\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}R\right)}{\left(\sqrt{\frac{ik}{\kappa}}R\right)^{\frac{1}{2}}}\right\}, \quad . \quad . \quad . \quad (41)$$

where r is the radius drawn from the original point source at a distance x_0 in front of the plane, r' is the radius drawn from the image point at distance x_0 behind the plane, and R is the radius drawn from a point at distance $x_0+\xi$ behind the plane, to any point on the same side of the plane as the original point source. The point sources associated with the reflected wave-system all lie on a line perpendicular to the plane passing through the original point source, and the expressions representing them all contain the functions

$$\left\{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}r\right)\Big/\left(\sqrt{\frac{ik}{\kappa}}r\right)^{\frac{1}{2}}\right\}$$

as characteristic functions expressing the manner in which they vary with respect to r, as is evident from (40), or can be made evident by expanding the terms in (41) by means of the addition theorem. From the form of the solution (41) it is clear that we can obtain the solution corresponding to an instantaneous point source of strength q instead of the periodic point source by an integration already performed in (13)—that is, we obtain the well-known solution

$$v = \frac{q\kappa}{8K(\pi\kappa t)^{\frac{3}{2}}}\left\{e^{-\frac{r^2}{4\kappa t}} + e^{-\frac{r'^2}{4\kappa t}} - \frac{2h}{K}\int_0^\infty d\xi\, e^{-\frac{h}{K}\xi - \frac{R^2}{4\kappa t}}\right\},$$

where . . . (42)

$$R^2 = (x+x_0+\xi)^2 + y^2 + z^2 \quad \text{and} \quad r'^2 = (x+x_0)^2 + y^2 + z^2,$$

the coordinates being referred to an origin at the point where the line of sources intersects the plane.

The corresponding results for a periodic or instantaneous

line source in the neighbourhood of a plane boundary can be reached by an exactly similar investigation based on the solutions referring to a cylinder of finite radius given in the paper referred to above.

The advantage of a point-source or line-source solution for the wave-system reflected from the plane is obvious when the complete plane is operating as a boundary. The question arises—can we make use of this solution when the plane boundary is incomplete or when only an exceedingly small portion of a plane is under consideration? If temperature effects are to be regarded as emanating radially from certain point sources to represent effects reflected, say, from a small element of a plane boundary, it would seem that discontinuities of temperature would exist at certain surfaces, and the applicability of the above results would require investigation. In general they would not be applicable to determine the effect of small portions of a plane boundary; but it might be possible, for instance, to solve the problem of a point source within a cylindrical boundary by considering the cylinder as formed of a succession of narrow plane strips, and by determining the effect of each strip by means of the above solution. What is required is to express the effect due to the whole boundary plane in terms of heat and temperature sources distributed over its surface, so that each element of area contributes a known quota to the reflected wave-system. The question raised above relates to the diffraction of heat, which opens up a wide range of problems for investigation, to which reference is made later. It is of fundamental importance, in relation to the solution of practical problems * arising in the laboratory, and must be dealt with later. The discussion of temperature sources by the method of the present paper must also be deferred, although their effects must be fully investigated before diffraction problems can be undertaken.

All the problems similar to those already discussed, relating to a point source within an infinite cylinder and in the neighbourhood of an infinite plane, can likewise be solved by a straightforward application of the method already used in the case of the spherical boundary. Expand the initial point-source solution by means of the addition theorem, so that each constituent represents a wave incident normally upon the boundary, and determine the boundary coefficient for each negative or reflected wave by means of the special boundary conditions. In this way a solution

* See "A Survey of Heat Conduction Problems," by Ezer Griffiths, Proc. Phys. Soc. p. 41 (1928-29).

equivalent to (41) for the plane boundary can be obtained, and a comparison of the two equivalent solutions should yield evaluations for a group of integrals. The discussion of these solutions for the cylinder and for the plane must, however, also be reserved for a later opportunity.

As an example of spherical wave-trains traversing the space between two boundaries we may take the problem of determining the temperature at any point within the space whose inner boundary is the sphere $r=b$, and whose outer boundary is the sphere $r=a$, due to a periodic point source of strength qe^{ikt} situated at any point O$'$ whose position is given by $r=a'$, $\theta=0$, between the boundaries, both boundaries being maintained at zero temperature. The wave-system diverging from O$'$, which we have formerly represented by v_1, where v_1 is expressed by (23) at all values of r exceeding $r=a'$, and by the same expression with r and a' interchanged at all values of r less than $r=a'$, now impinges on both boundaries. The latter expression represents the wave-system incident upon the inner boundary. Each positive constituent of the wave-system represented by a term of the type

$$\alpha\, e^{ikt}\, \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)\Big/ r^{\frac{1}{2}},$$

after reaching boundary $r=a$, continues as a negative wave-train represented by

$$\mathrm{A}_m\, \alpha\, e^{ikt}\, \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)\Big/ r^{\frac{1}{2}},$$

where A_m is the boundary coefficient or operator at $r=a$. Similarly each negative wave-train represented by a term of the type

$$\beta\, e^{ikt}\, \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)\Big/ r^{\frac{1}{2}},$$

after reaching boundary $r=b$, continues as a positive train represented by

$$\mathrm{B}_m\, \beta\, e^{ikt}\, \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)\Big/ r^{\frac{1}{2}}.$$

where B_m is the boundary coefficient or operator at $r=b$. From the consideration of successive reflexions, as explained in detail in a previous paper, we find that the wave-system built up between the two boundaries consists of

$$(\alpha+\mathrm{B}_m\beta)\,\{1+\mathrm{A}_m\mathrm{B}_m+\mathrm{A}_m^2\mathrm{B}_m^2+\mathrm{A}_m^3\mathrm{B}_m^3+\text{ etc.}\}$$
$$e^{ikt}\,\mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right)\Big/ r^{\frac{1}{2}}, \quad . \quad . \quad (43)$$

and $(\beta+A_m\alpha)\{1+A_m B_m+A_m^2 B_m^2+A_m^3 B_m^3+\text{ etc.}\}$

$$\epsilon^{ikt}\, I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,r\right)\Big/ r^{\frac{1}{2}}, \quad . \quad . \quad . \quad (43')$$

as the sum of positive waves and sum of negative waves respectively arising from one constituent of the diverging system. These contain the complete solution required for one constituent, provided the negative trains do not transmit effects across the boundary $r=b$ which involve the consideration of wave-trains returning to the boundary $r=b$ after passing through the origin. This condition is fulfilled in the present case, and from (23) and (25) it appears that the values of the coefficients required are given by

$$A_m=-\frac{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)} \quad \text{and} \quad B_m=-\frac{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)}{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)} \quad . \quad . \quad . \quad (44)$$

$$\left.\begin{aligned}
\alpha+B_m\beta &= \frac{q}{2\pi K}\frac{1}{(a')^{\frac{1}{2}}}(m+\tfrac{1}{2})\,P_m(\cos\theta)\left\{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right)\right. \\
&\qquad \left. -K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right)\frac{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)}{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)}\right\}, \\
\beta+A_m\alpha &= \frac{q}{2\pi K}\frac{1}{(a')^{\frac{1}{2}}}(m+\tfrac{1}{2})\,P_m(\cos\theta)\left\{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right)\right. \\
&\qquad \left. -I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a'\right)\frac{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}{I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}\right\}, \\
\frac{1}{1-A_m B_m} &= \frac{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}{K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)-K_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)I_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)}.
\end{aligned}\right\} \quad . \quad . \quad . \quad (44)$$

The complete solution of the problem proposed may therefore be written in the form

$$v = \frac{\dot{q}}{2\pi \mathrm{K}} \frac{e^{ikt}}{(ra')^{\frac{1}{2}}} \sum_{m=0}^{m=\infty} (m+\tfrac{1}{2}) \mathrm{P}_m(\cos\theta)$$

$$\left\{ a_m \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right) + b_m \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, r\right) \right\}, \qquad (45)$$

where

$$\left.\begin{aligned} a_m &= \left[\mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, b\right) \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) - \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, b\right)\right] \Big/ d_m \, \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right), \\ \text{and} \\ b_m &= \left[\mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right) \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) - \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a'\right) \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right)\right] \Big/ d_m \, \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, b\right), \\ \text{and} \\ d_m &= \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, b\right) \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right) - \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, a\right) \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\, b\right). \end{aligned}\right\} \quad (45')$$

It is to be noted that we have included the original wave-system diverging from the point source at $r=a'$ as the first positive term in (43) for $r>a'$, and as the first negative term in (43′) for $r<a'$.

If in the above process we simply alter the values of the constants A_m and B_m to the values given by (29) and (29′) for A_m, and by the corresponding expression, with r and a' interchanged and b in place of a for B_m, A_m and B_m being each determined by (27), we obtain the solution to the problem in which radiation occurs at both boundaries. The above solution applies to any specified boundary conditions

with the values of A_m and B_m corresponding to these conditions inserted in place of the values given in (44) above, always provided the boundary conditions at the inner boundary do not involve the consideration of wave-trains continuing by transmission across the boundary $r=b$ and returning to it after passing through the origin. Such a condition would arise if conduction of heat took place to an inner sphere at the inner boundary. The modifications of the above solution for such a case are discussed in the preceding paper *.

Moreover, the above solution, modified in exactly the manner described on pp. 241, 242 above, applies immediately to the corresponding problem of a line source of strength qe^{ikt} per unit length, situated at a position given by $\rho=a'$, $\theta=0$, in the space between two infinite cylinders of radii $\rho=b$ and $\rho=a$ respectively.

Analogies.

The investigations contained in the present paper and its predecessors make evident the utility of the method of wave-trains in arriving at the solution of heat-conduction problems. It is well known, however, that exactly similar differential equations govern a large number of physical processes involving diffusion. Examples of such are diffusion of ions in gases and of dissolved substances in solution, diffusion of electric currents in conductors, diffusion of shots round a target, diffusion of potential in wave motion, and of motion throughout a viscous liquid, and many others. Problems analogous to those dealt with above arise in all these subjects, and solutions of all problems in these subjects can be developed from the same positive and negative wave-trains. We have also the well-known analogies existing between the mathematical theory of steady conduction of heat and the mathematical theory of potential in a variety of subjects. In potential theory we have flow of force, where in heat-conduction theory we have flow of heat. The analogy exists in all these cases because there is a fundamental similarity in the physical processes with which these different subjects are concerned ; but if we can represent the process of heat conduction among others by a system of wave-trains, additional analogies and similarities arise in diverse subjects out of the fact that mathematically these subjects come within the scope of the general subject of wave-motion, and the treatment of all problems involving certain specified boundaries is the same for them all. As

* Phil. Mag. (Feb. 1930).

an example of the manner in which solutions obtained for problems in heat may be employed to provide corresponding solutions for problems in other subjects, and conversely, consider the differential equation for the potential function

$$\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{\partial^2 V}{\partial z^2}=0. \quad . \quad . \quad . \quad (46)$$

When we confine our investigation to solutions of the type e^{ikz}, the equation to determine V as a function of ρ becomes

$$\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}-k^2 V=0, \quad . \quad . \quad . \quad . \quad (47)$$

from which it appears that the two fundamental positive trains and the two fundamental negative trains are given by

$$V=A\,e^{ikz}\,K_0(k\rho) \quad \text{and} \quad V=B\,e^{ikz}\,I_0(k\rho) \text{ respectively}, \quad (48)$$

these being similar to the fundamental wave-trains diverging from or converging to a line source of heat. In these z replaces t, and $k\rho$ in each Bessel function replaces $\sqrt{\frac{ik}{\kappa}}\rho$ in the corresponding heat solutions. The difficulty of finding a physical interpretation of these wave-trains remains, although, by means of the Lorentz transformation applied to the equation (48) above, we can obtain a group velocity representing the actual velocity of matter.

Suppose now we take the solution of (46) representing unit mass of matter situated at the origin. It is given by

$$V=\frac{2}{\pi}\int_0^\infty dk\,e^{ikz}\,K_0(k\rho)=\frac{1}{r}, \quad . \quad . \quad . \quad (49)$$

in which only the diverging waves appear. The corresponding heat solution is given by

$$v=\frac{q}{2\pi^2 K}\int_0^\infty dk\,e^{ikt}\,K_0\left(\sqrt{\frac{ik}{\kappa}}\rho\right), \quad . \quad . \quad (50)$$

which represents an instantaneous line source of strength q per unit length. Proceed next to find the Green's function for the space within a cylinder of radius a in the potential problem corresponding to a pole at $(\rho=a', z=z')$. V is to be zero at the surface of the cylinder and have the value $1/r'$ near the pole. The solution of the potential problem is

$$V=\frac{2}{\pi}\int_0^\infty dk\,e^{ik(z-z')}\sum_{m=-\infty}^{m=\infty}\left\{K_m(k\rho)\,I_m(ka')\right.$$
$$\left.-K_m(ka)\,I_m(ka')\frac{I_m(k\rho)}{I_m(ka)}\right\}\cos m(\theta-\theta'), \quad (51)$$

obtained from (48) and (49) by exactly the same reasoning as the corresponding heat solution given in equation (70) of a former paper (Phil. Mag., April 1928) is obtained from equation (50) above. Similarly, by analogy with the solution given in (45), modified so as to apply to two cylindrical instead of to two spherical boundaries, we obtain the Green's function for the space between two cylinders of radii $\rho=b$, $\rho=a$ respectively in the form

$$V = \frac{2}{\pi}\int_0^\infty dk\, \epsilon^{ik(z-z')} \sum_{m=-\infty}^{m=+\infty} \{a_m K_m(k\rho) + b_m I_m(k\rho)\} \cos m(\theta-\theta'), \quad . \quad . \quad (52)$$

where a_m and b_m are given in (45′) above with the Bessel functions all altered to order m and $\sqrt{\frac{ik}{\kappa}}$ replaced throughout by k.

Consider next, as an example of a different type, the differential equation associated with the propagation of waves

$$\frac{\partial^2\phi}{\partial t^2} = V^2 \nabla^2\phi, \quad . \quad . \quad . \quad . \quad . \quad (53)$$

where ϕ may be taken for the purpose of illustration as the velocity potential of the motion of particles in transmitting sound waves. If we consider wave motion symmetrical about a line as in the equation (1) of the present paper, and confine our investigation to cases in which ϕ varies according to ϵ^{ikt}, the equation to determine ϕ as a function of r and θ becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{k^2}{V^2}\phi + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) = 0, \quad (54)$$

which differs from (2) in one term only. Corresponding to the two solutions of (1) given in (5), we have in this case

$$\phi = A\,\epsilon^{ikt}\, G_{m+\frac{1}{2}}\left(\frac{k}{V}r\right) P_m(\cos\theta)/r^{\frac{1}{2}}$$

$$\text{and} \qquad \phi = B\,\epsilon^{ikt}\, J_{m+\frac{1}{2}}\left(\frac{k}{V}r\right) P_m(\cos\theta)/r^{\frac{1}{2}}, \quad . \quad (55)$$

where the Bessel functions that appear are connected with those appearing in (5) by the relations

$$G_m\left(\frac{k}{V}r\right) = K_m\left(\sqrt{\frac{i^3k^2}{V^2}}\,r\right) \text{ and } J_m\left(\frac{k}{V}r\right) = I_m\left(\sqrt{\frac{i^3k^2}{V^2}}\,r\right) \quad . \quad . \quad . \quad (56)$$

Consider now the solution of (53) representing a source of sound at the origin, for which the motion of fluid is symmetrical about the origin. The required solution involves only the diverging wave, and must be represented by

$$\phi = \mathrm{A}\, e^{ikt}\, \mathrm{G}_{\frac{1}{2}}\left(\frac{k}{\mathrm{V}}r\right)\Big/ r^{\frac{1}{2}}, \quad \ldots \quad (57)$$

where A is determined by the condition that

$$\underset{r\to 0}{\mathrm{Lt}}\left(4\pi r^2 \frac{\partial\phi}{\partial r}\right) = e^{ikt}. \quad \ldots \quad (58)$$

Thus in parallel with the heat solution for a point source we have the solution for a point source of sound in the form

$$\phi = \frac{1}{4\pi}\sqrt{\frac{2}{\pi}\frac{k^2}{\mathrm{V}^2}}\, e^{ikt}\, \mathrm{G}_{\frac{1}{2}}\left(\frac{k}{\mathrm{V}}r\right)\Big/\left(\frac{k}{\mathrm{V}}r\right)^{\frac{1}{2}}. \quad . \quad (59)$$

Compare this with (9) above. The Bessel function $\mathrm{G}_{\frac{1}{2}}$ replaces the corresponding function $\mathrm{K}_{\frac{1}{2}}$, and $\frac{k}{\mathrm{V}}$ replaces $\sqrt{\frac{ik}{\kappa}}$. Proceed now to consider the effect of a rigid boundary. The condition to be fulfilled in the case of a problem in sound is that there is no transmission of fluid across such a boundary. The corresponding condition in a heat problem is that there is no transmission of heat. If we introduce this condition, for instance, in the problem of a point source situated between two spherical surfaces $r=b$ and $r=a$ respectively, we must take

$$\left.\begin{aligned} \mathrm{A}_m &= -\frac{2\sqrt{\frac{ik}{\kappa}}\,a\,\mathrm{K}'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right) - \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}{2\sqrt{\frac{ik}{\kappa}}\,a\,\mathrm{I}'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right) - \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,a\right)}\,; \\ \mathrm{B}_m &= -\frac{2\sqrt{\frac{ik}{\kappa}}\,b\,\mathrm{I}'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right) - \mathrm{I}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)}{2\sqrt{\frac{ik}{\kappa}}\,b\,\mathrm{K}'_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right) - \mathrm{K}_{m+\frac{1}{2}}\left(\sqrt{\frac{ik}{\kappa}}\,b\right)} \end{aligned}\right\} \quad \ldots \quad (60)$$

instead of the values given in (44), and modify (45) in accordance with this. Equation (45), so modified, will then be the solution of the heat problem. Convert each Bessel function it contains, each K_m into G_m, and each I_m into J_m,

replacing $\sqrt{\frac{ik}{\kappa}}$ in each by $\frac{k}{\mathrm{V}}$, and the equation will then represent the solution to the corresponding problem in sound.

If the boundary with which we are concerned in a sound problem is that separating two gases, we have the conduction of sound across the boundary governed by the conditions (for the case of a spherical surface)

$$\frac{\partial\phi}{\partial r_i}=\frac{\partial\phi'}{\partial r_0}, \qquad (\rho\phi)_i=(\rho'\phi')_0, \quad . \quad . \quad (61)$$

the suffixes indicating the inner side and outer side respectively of the boundary. The corresponding heat conditions are

$$\mathrm{K}_1\frac{\partial v_1}{\partial r}=\mathrm{K}_2\frac{\partial v_2}{\partial r} \quad \text{and} \quad v_1=v_2, \quad . \quad . \quad . \quad (62)$$

which clearly indicate similar mathematical conditions, so that we may look for similarities in solutions relating to the same boundaries. Analogies similar to the above exist in various wave-motion subjects. The matter will repay further investigation, and should lead to the solution of important practical problems involving diffraction of heat by obstacles of various forms, and also to the solution of many problems in conduction of heat by the aid of analogous solutions obtained for problems in other departments of mathematical physics.

It will be seen from the above discussion that the solution of problems by means of wave-trains introduces a great unification throughout mathematical physics. In the special application of this method to heat problems the determination of effects due to continuous sources has not yet been carried out, and it is desirable that this should be done in view of its application to many problems of steady heat supply.

XIX. *Some Magnetic Alloys and their Properties.* *By* H. H. Potter, *Ph.D., Lecturer in Physics, University of Bristol**.

Part I.—The System Ag–Mn–Al.

One of the most striking physical properties of the element manganese is its tendency to form in conjunction with other non-ferromagnetic elements binary and

* Communicated by the Author.

ternary alloys which show pronounced ferromagnetism *. On account, presumably, of their superior mechanical properties the ternary alloys, particularly those with Cu and Al † and Cu and Sn ‡, have received more attention than the binaries. As far, however, as the writer is aware all the strongly ferromagnetic ternary alloys which have so far been examined in detail contain Cu as well as Mn, although the third element may be either Al, Sn, As, Sb, Bi, or B. There is good reason to believe that the manganese is the essential factor producing the ferromagnetism, the other elements serving to produce the correct crystal structure and possibly causing some modification of the electron distribution in the Mn atom favourable to the manifestation of its ferromagnetic properties. One would expect therefore to be able to find other ferromagnetic ternary alloys of Mn not containing Cu. On account of its position relative to Cu in the periodic table the most obvious element to try in this respect is silver §.

Preparation and Analysis of Alloys.

Ternary alloys of Ag, Mn, and Al were prepared by melting Ag and Mn together under $BaCl_2$ in a gas furnace. The melt was then divided into portions and varying amounts of Al added. The constituents used were of high purity; the Ag was supplied by Messrs. Johnson Matthey, the Al by the British Aluminium Co., and the Mn by Kahlbaum. The principal impurity was present in the Mn, which was about 99 per cent. pure.

Owing to the tendency of these alloys to form scale the resultant constitution did not agree with the amounts of the elements originally used, and chemical analysis of the specimens was necessary. The silver was estimated by titration with ammonium sulphocyanide. The Mn content was determined by oxidizing a nitric acid solution with sodium bismuthate and estimating the permanganic acid with ferrous ammonium sulphate. The aluminium was estimated by precipitation with ammonia and ignition of

* For a bibliography see O. v. Auwers, *Jahrbuch der Rad. u. Elektronik*, xvii. p. 181 (1920).

† *Ibid.* p. 186.

‡ Take and Semm, *Verh. d. Phys. Ges.* xvi. p. 971 (1914).

§ Since this work was started the writer has seen a British Patent, No. 169,144 (1921), taken out by the Isabelle Hütte Ges., for the use of Ag, Mn, Al alloys as a substitute for table silver. It is mentioned in this patent that the alloys are ferromagnetic, but no details are given.

the precipitate to Al_2O_3. Great care was taken to get the precipitate free from manganese *. In almost all cases the total of Ag, Mn, and Al added up to the total weight of alloy to within 0·3 per cent.†

Magnetic Measurements.

The magnetic properties of these alloys were examined by means of the ballistic method. The material, which is very easily workable, was turned in the lathe into specimens 9 mm. long and 3 mm. diameter. These were placed in a search coil between the poles of an electromagnet capable of giving a field of 20,000 gauss with a gap of 11 mm. The field lines were carefully compensated by using a second search coil in series with the first and mounted close to it. Owing to the appreciable time taken to establish the field of the electromagnet the galvanometer with which the flux was measured was loaded to have a period of about 15 secs Absolute values of the intensities were obtained by calibration of the system with a specimen of pure nickel, the saturation intensity of which was taken as 500 units per c.c.

Saturation Intensity and Chemical Constitution.

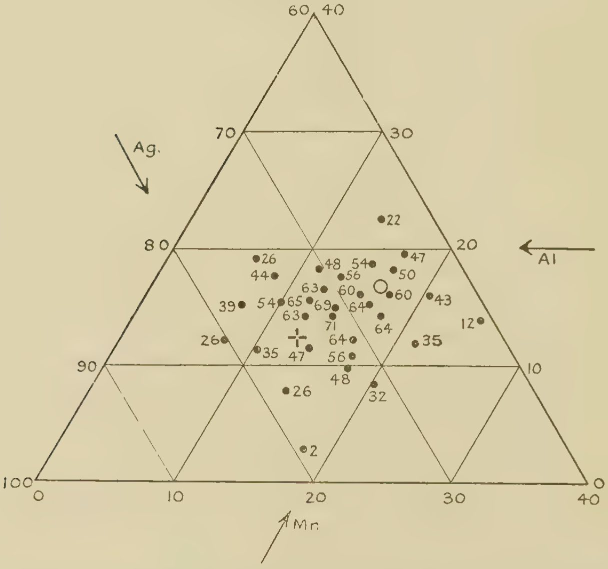
The results of measurements on saturation intensity are shown in fig. 1. The chemical constitution, expressed in atomic percentages, is represented by points on an ordinary triangular diagram, the number associated with any point giving the volume intensity of magnetization of the particular alloy represented by the point. The intensities were measured in a field of 20,000 gauss. The alloys all received the same heat treatment, first being cast into carbon moulds and then annealed for some days—until the magnetizability no longer increased—at 250° C. This method has been found to produce the most magnetic specimens. Alloys slowly cooled from the melting-point are almost non-magnetic, whilst annealing at temperatures above 250° C. produces a deterioration of magnetic properties which is more rapid the higher the annealing temperature. For instance, annealing for one hour at 850° C., followed either by sudden or slow cooling, almost completely destroys the magnetic properties.

* See Lundell and Knowles, Journ. Amer. Chem. Soc. xlv. p. 676 (1923).

† My best thanks are due to Dr. T. Malkin, of the Chemistry Department of this University, for much valuable advice in the chemical analysis.

The most strongly magnetic alloy was at first thought to have a constitution corresponding to about four atoms of Ag to one each of Mn and Al. More detailed analysis of the system, however, has shown that better results are obtained with an alloy containing about five atoms of Ag to one each of Mn and Al. The maximum intensity considered as a function of chemical constitution is certainly rather flat, and some little doubt may be felt about its exact position on account of the difficulty of attaining saturation with these

Fig. 1.



The numbers represent the saturation intensities per c.c. The positions of the points give the atomic percentages of the constituents.

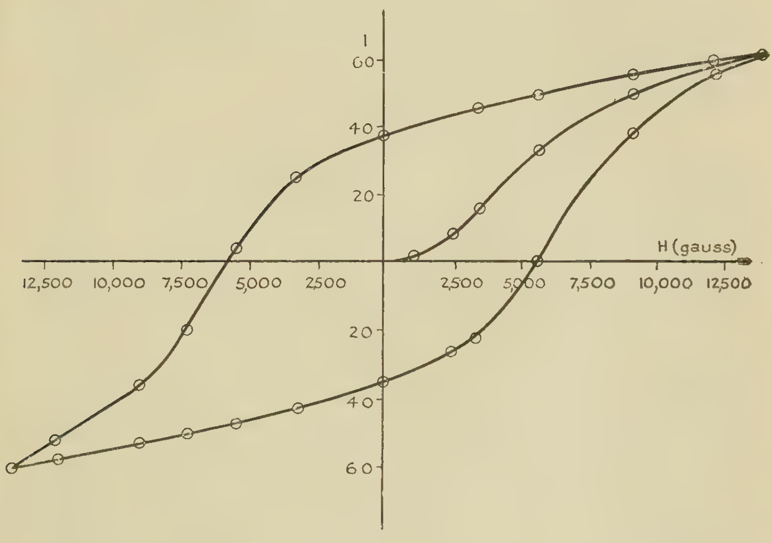
alloys. The maximum does, however, appear to be much nearer the constitution Ag_5MnAl than to any other simple proportions, although it is not suggested that a compound of this constitution actually exists. The best alloy prepared, which in fig. 1 is marked 71, has a constitution 86·9 per cent. Ag, 8·8 per cent. Mn, 4·3 per cent. Al, or, expressed in atomic percentages, 71·5 per cent. Ag, 14·3 per cent. Mn, and 14·2 per cent. Al. This is almost exactly represented by the formula Ag_5MnAl. The way in which the saturation

intensity of other alloys falls off the farther one departs in any direction from this point also suggests that this is the position of the maximum rather than either Ag_4MnAl (marked with a circle) or Ag_6MnAl (marked with a cross).

Coercive Forces and Hysteresis Loss.

The most remarkable property of these alloys is their enormous coercive force. A typical hysteresis curve is shown in fig. 2. The actual values of remanence and

Fig. 2.



coercive force vary from alloy to alloy, but for the most magnetic alloys the latter is about 5000 gauss. In some cases it is considerably more than this, and in most cases is slightly increased by annealing.

This enormous coercive force appears not only to be greater than anything previously observed, but to be even of a different order of magnitude. The Heusler alloys, for instance, have a coercivity varying from a few gauss to two or three hundred gauss according to the heat treatment, whilst the hardest permanent magnet steels have a coercive force of about 250 gauss *.

* Honda and Saitô, *Sci. Rep. Tohoku*, ix. p. 417 (1920).

The energy loss per cycle may be considerably more than 10^6 ergs per c.c. In spite of the low value of the saturation intensity this represents a hysteresis loss greater than that in the hardest steels.

Curie Points.

For the most magnetic of these alloys the temperature at which the ferromagnetism vanishes is about 360° C. It is well defined and shows little or no temperature hysteresis.

X-ray Structure.

An X-ray examination of the alloys by the powder method shows them to possess a face-centred cubic structure with lattice constant of about 4·06 Å.U. This is identical within experimental error with the structure of pure silver. No superstructure has been detected—in fact it is difficult to see how seven atoms could be fitted into a superstructure in a cubic system. A similar difficulty exists, of course, in the case of permalloy in which five atoms—four of nickel and one of iron—have to be fitted into a similar structure *. It would therefore appear necessary to regard the alloy as a homogeneous solid solution of Mn and Al in Ag. In this respect it differs from the Heusler alloys. These latter were at first thought to be a solid solution of Mn_3Al in Cu_3Al, but since recent X-ray examination † has revealed the existence of a simple superstructure, it would appear more reasonable to attribute the ferromagnetic properties of the Heusler alloys to the compound Cu_2MnAl, in which, however, both Cu and Mn are soluble.

The fact that the Ag–Mn–Al alloys show no superstructure has removed the possibility of carrying out one of the original objects of the research. It has been shown ‡ that the directional magnetic properties of single crystals of Heusler alloys are probably determined entirely by the positions of the Mn atoms, but this could not be definitely proved on account of the difficulty of distinguishing by means of X-rays between Mn and Cu. It was hoped that by using a much heavier element like Ag instead of Cu a more complete solution of the crystal structure might have been obtained. In view, however, of the alloy appearing to be homogeneous there is no point in proceeding to single crystal work.

* L. McKeehan, Journ. Franklin Inst. cciv. p. 501 (1927).
† E. Persson, *Zeits. f. Phys.* lvii. p. 115 (1929).
‡ Potter, Proc. Phys. Soc. London, xli. p. 135 (1929).

Part II.—The Binary System Sn–Mn.

Since the Al in the system Cu–Mn–Al (the Heusler alloys) may be replaced by Sn without loss of ferromagnetic properties, it seemed profitable to examine the result of substituting Sn for Al in the system Ag–Mn–Al. The system Ag–Mn–Sn * was immediately shown to possess marked ferromagnetism, which was found, however, to be due entirely to binary alloys of Sn and Mn, the Ag merely acting as a dilutant. This system has been examined previously by R. S. Williams †, Fassbender ‡, and Honda §. According to the former there exist in this system three compounds—$SnMn_4$, $SnMn_2$, and SnMn. All three observers find a maximum intensity of magnetization at a manganese content which corresponds to $SnMn_4$. Williams further states that $SnMn_2$ is also magnetic and SnMn non-magnetic.

Preparation of Alloys and Analysis.

For the present work alloys were prepared by melting together pure Sn (kindly presented by Messrs. Capper Pass) and Kahlbaum Mn in a gas furnace. The melt was cast into rods which could be ground down to specimens of the right size for magnetic tests by the ballistic method (see Part I. of this paper). The alloys—particularly those containing from 35 to 55 per cent. of Mn—were extremely brittle, and it was quite out of the question to turn specimens in the lathe; Fassbender in fact found the alloys so difficult to work with that he used them in powder form. We have, however, been able to make suitable specimens by using a very sharp grindstone.

The estimation of the Mn content was carried out by the method employed in Part I. of this paper. The Sn was estimated by conversion into metastannic acid and ignition to oxide.

Magnetic Intensity Measurements.

The most magnetic alloys are prepared by casting and subsequently annealing at 450° to 500° C. Slow cooling from the melting-point or annealing at temperatures above 500° C. results in a lowering of the saturation intensity.

* Many of the results on these alloys were obtained in collaboration with Mr. H. Harbour, to whom my best thanks are due.

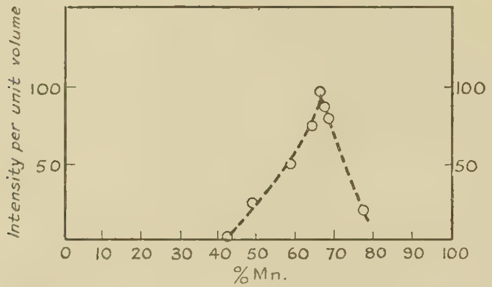
† Williams, *Zeits. f. anorg. Chemie*, lv. p. 24 (1907).

‡ Fassbender, *Verh. d. Deutsch. Phys. Ges.* x. p. 260 (1908).

§ Honda, *Ann. d. Phys.* xxxii. p. 1024 (1910).

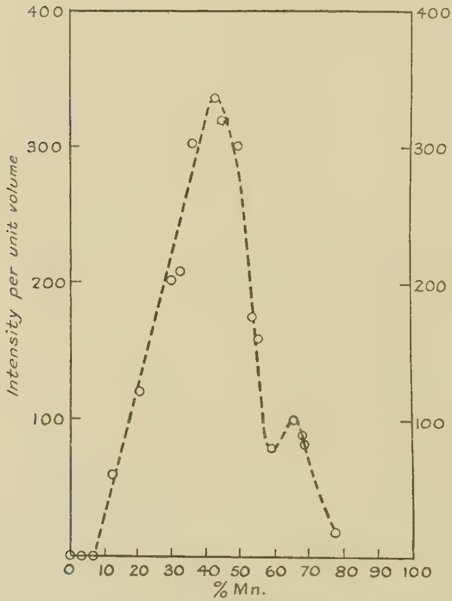
The results of measurements of saturation intensities before annealing is shown in figs. 3 and 4. Fig. 3 shows

Fig. 3.



Saturation intensity at room temperature as a function of percentage of Mn.

Fig. 4.



Saturated intensity at the temperature of liquid air as a function of the percentage of Mn.

the saturation intensity at room temperature plotted as a function of the Mn content of the alloy, while in fig. 4 the

same is done for liquid-air temperatures. The difference in these two curves shows clearly the existence of two ferro-magnetic substances, the one with a high saturation intensity and low Curie point, the other with a higher Curie point and much lower saturation intensity. The room-temperature magnetization shows a maximum at 65 per cent. Mn corresponding to the compound $SnMn_4$. The other curve shows a maximum at 42 per cent. Mn. The compound $SnMn_2$ contains 48·5 per cent. of Mn, but examination of the equilibrium diagram for the system shows that, even if $SnMn_2$ is responsible for the high magnetic intensity which is found at liquid air temperatures, the maximum in cast alloys would not be expected at 48·5 per cent. Mn. The equilibrium diagram has been partially worked out by Williams *. According to his results a melt containing 48·5 per cent. Mn would separate out as $SnMn_4$ until the Mn content was reduced to 42 per cent., after which $SnMn_2$ would settle out. Owing then to the presence of $SnMn_4$ we should expect the maximum magnetic intensity to be found not at 48·5 per cent. Mn, but more nearly at 42 per cent. Mn. It is significant too that alloys with less than 42 per cent. Mn are not magnetic at room temperatures, since below this Mn content the alloy should contain no $SnMn_4$. The absence of ferromagnetism even at liquid air temperatures in alloys containing less than 7 per cent. Mn is also in agreement with Williams's diagram, since below this Mn content no $SnMn_2$ is formed. Whereas, however, Williams speaks of $SnMn_4$ as being more magnetic than $SnMn_2$, we now find that by working at liquid air temperatures the saturation intensity of $SnMn_2$ is much the greater, and by suitable annealing may be made nearly equal to that of pure nickel.

The results of annealing are very complicated, but alloys containing less than 35 per cent. Mn deteriorate rapidly on annealing at 450° to 500° C. This might be explained on account of the fact that annealing at this temperature would probably convert some of the $SnMn_2$ originally present into the non-magnetic SnMn.

Alloys with a Mn content greater than 35 per cent. improve on annealing at 500° C., but annealing at higher temperatures causes the magnetization to decrease.

Curie Points.

The Curie points of both the magnetic compounds $SnMn_2$ and $SnMn_4$ are rather ill-defined, the magnetization not

* Williams, *loc. cit.*

approaching so rapidly to zero with rising temperature as is usually the case with ferromagnetic substances. The respective two Curie points are estimated roughly as 0° C. and 150° C.

Summary.

Part I.

It has been shown that the ternary system Ag–Mn–Al possesses ferromagnetic properties with a maximum saturation intensity of about 70 units per c.c. at or near the point represented by five atoms of Ag, one of Mn, and one of Al. The alloys of this system are therefore less magnetic than those of the Cu–Mn–Al system (the Heusler alloys), which have at the best a saturation intensity of over 400 units per c.c.

The new alloys have a Curie point of about 360° C. and an enormous coercive force, which in the most magnetic specimens is about 5000 gauss, but which in many cases considerably exceeds this figure.

Part II.

Two magnetic compounds are found in the binary system Mn–Sn. The first, $SnMn_4$, has a saturation intensity of about 100 units and a Curie point at about 150° C. The second compound—presumably $SnMn_2$—has been obtained after suitable annealing with a saturation intensity as high as 470 units/c.c., and has a Curie point at about 0° C. In general ferromagnetic substances having high saturation intensities have also high Curie points, so that $SnMn_2$ would appear to be unique in having at the same time a low Curie point and a saturation intensity almost equal to that of nickel. The low Curie point of this compound has resulted in previous workers entirely missing its very strong ferromagnetism.

Acknowledgments.

The work was carried out in the Henry Herbert Wills Physical Laboratory, University of Bristol, and my best thanks are due to my colleague, Mr. Sucksmith, for much helpful discussion, and to the Colston Research Society of this University for a grant towards the expenses of this investigation.

XX. *Ballistic and Perfect Balances in Bridges treated by the Operational Calculus.* By A. T. STARR, *B.A.*, *B.Sc.*

(International Telephone and Telegraph Labs. Inc.) *.

Introduction.

THE conditions for a ballistic balance in bridges has always seemed a matter of some complexity and length. This paper describes a simple method for finding the conditions for perfect or ballistic balance, whereby much work is saved. The method is proved by the operational calculus. Also the use of inverse networks in forming a bridge with a perfect balance is given, and two types of bridges—the equal ratio bridge and the inverse network bridge—are discussed.

Some theorems on inverse networks and equal ratio bridges are given for general and ballistic balances.

Determination of a Balance in Steady State Bridges.

The usual method of determining a bridge balance in steady-state currents is to put $j\omega L$, R, $\frac{1}{j\omega C}$ for the impedance of inductance coil L, resistance R, condenser of capacity C respectively, find the current in the detector, and equate its real and imaginary parts to zero. This will give a balance only when the current is of pulsatance ω. The d.c. case is, of course, given by putting $\omega=0$.

If, however, the relations between the bridge elements for a balance are independent of the frequency the balance is perfect, *i. e.*, an e.m.f. of any wave shape will cause no current to flow in the detector. This is proved by superposition. This way of determining a perfect balance is simple, but unfortunately it cannot be used for the case of a ballistic balance.

Determination of a Perfect or Ballistic Balance by the Method of the Operational Calculus.

In place of $j\omega L$, R, and $\frac{1}{j\omega C}$, we put pL, R, and $\frac{1}{pC}$ for the impedance of coil, resistance, and condenser respectively, where $p=\frac{d}{dt}$ and $\frac{1}{p}f(t)=\int^t f(t)\,dt$. The e.m.f. E$(t)$ in any

* Communicated by the Author.

part of the bridge then gives a current in the detector, which is found to be

$$i = f(p)\mathrm{E}(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$f(p)$ is the transfer admittance in operational form between the input terminals and the detector.

Let

$$f(p) = \frac{p^{-m}(a_0 + a_1 p + a_2 p^2 + \ldots)}{b_0 + b_1 p + b_2 p^2 + \ldots} \quad . \quad . \quad . \quad . \quad (2)$$

It is proved in the Appendix I. that for a perfect balance all the a's must vanish, *i. e.*,

$$a_0 = a_1 = a_2 = \ldots = 0. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For a ballistic balance only the first $(m+2)$ a's need vanish, *i. e.*,

$$a_0 = a_1 = \ldots = a_{m+1} = 0. \quad . \quad . \quad . \quad . \quad (4)$$

Some Examples of the Method.

As one example we will consider a modification of Maxwell's bridge, shown in fig. 1, given by Harris and Williams (P. O. E. E. April 1930, pp. 36–41). We consider only the case when there is no coupled secondary. The case of a coupled secondary is considered later.

The conditions for ballistic and perfect balances will now be found. $f(p)$ is found very simply to be of the form

$$f(p) = \frac{\frac{1}{\mathrm{C}}(\mathrm{SQ}-\mathrm{PR}) + p\left[\mathrm{SQ}(\mathrm{P}+\mathrm{T}) - \mathrm{PTR} - \frac{\mathrm{PL}}{\mathrm{C}}\right] - p^2\mathrm{PTL}}{b_0 + b_1 p + b_2 p^2},$$

where b_0, b_1, and b_2 are positive constants.

Therefore for a perfect balance

$$\left.\begin{aligned} \frac{1}{\mathrm{C}}(\mathrm{SQ}-\mathrm{PR}) &= 0, \\ \mathrm{SQ}(\mathrm{P}+\mathrm{T}) - \mathrm{PTR} - \frac{\mathrm{PL}}{\mathrm{C}} &= 0, \\ \text{and} \qquad \mathrm{PTL} &= 0, \end{aligned}\right\}$$

giving

$$\mathrm{SQ} = \mathrm{PR} = \frac{\mathrm{L}}{\mathrm{C}} \quad \text{and} \quad \mathrm{T} = 0.$$

For a ballistic balance it is necessary only that the first two relations be satisfied, viz.,

$$\mathrm{SQ} = \mathrm{PR} = \frac{\mathrm{L}}{\mathrm{C}}.$$

This method can be applied with similar ease to all the cases treated in the above paper.

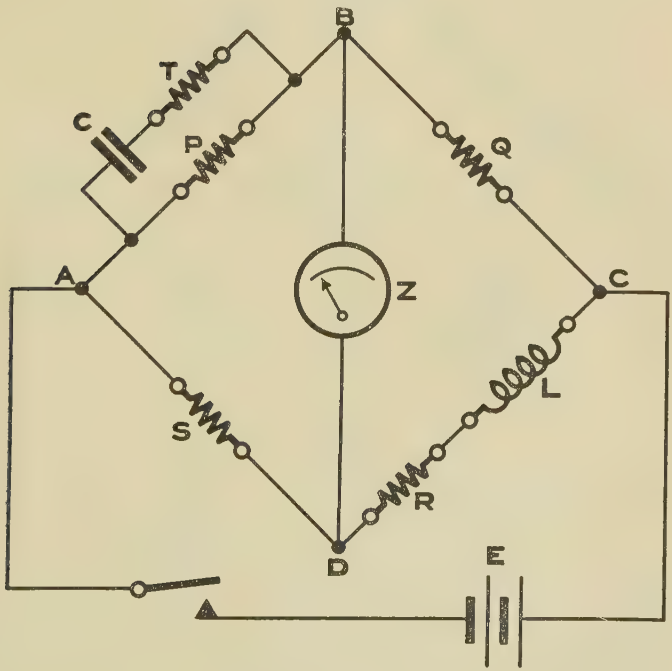
As another example consider the bridge of Vaschy and Pirani; this bridge is shown in fig. 2. The current in the detector is

$$\frac{Z_2Z_3 - Z_1Z_4}{AZ_1 + B},$$

where

$$Z_1 = P_1 + pL + \frac{P_2}{1 + pcP_2}, \quad Z_2 = Q, \quad Z_3 = R, \quad Z_4 = S,$$

Fig. 1.



and A and B are positive constants. A perfect balance is clearly unobtainable.

For a ballistic balance

$$\left[\frac{1}{p}(Z_2Z_3 - Z_1Z_4)\right] = 0 \quad \text{for } p = 0,$$

giving

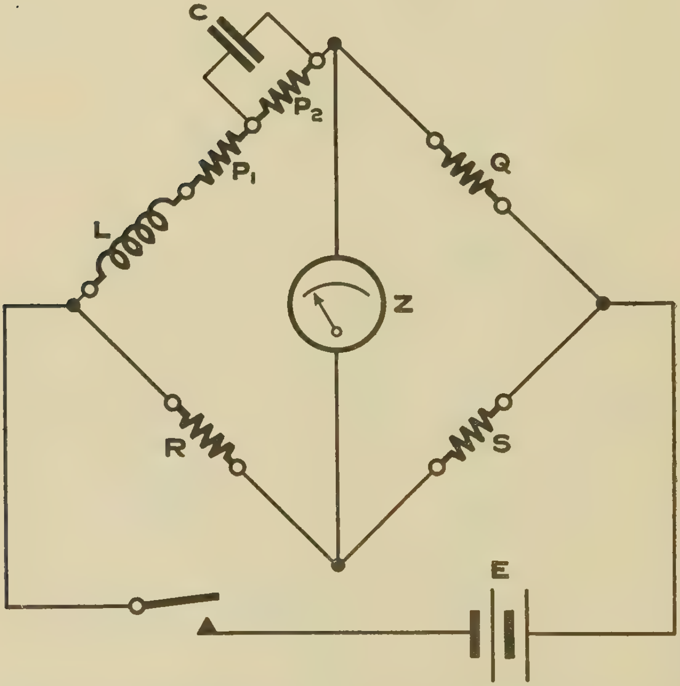
$$QR = S(P_1 + P_2) \quad \text{and} \quad L = CP_2^2.$$

It is unnecessary to find A and B, and the simplicity of the method is obvious.

Two important Classes of Bridges.

In the general bridge shown in fig. 3 there are two important general classes amongst others. In one class

Fig. 2.



$Z_1 = Z_2 =$ a known impedance, when the condition for a balance is

$$Z_3 = Z_4.$$

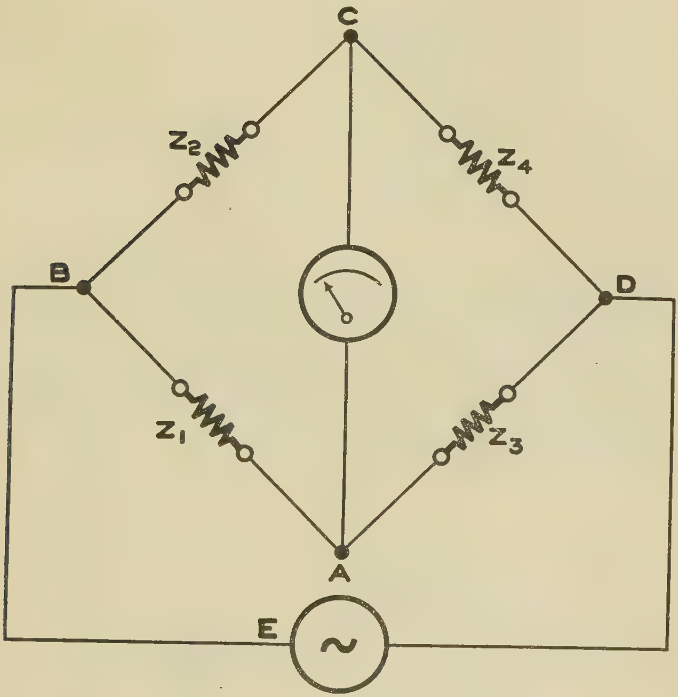
This equality may hold at all frequencies, as in the capacity and conductance bridge, or at one frequency only, as in the series-resonant bridge. The arms Z_1 and Z_2 are called the ratio arms, and are usually equal resistances; but they can be any two equal impedances whatsoever. Z_3 is built up out of variable standard impedance elements, parts of which may be put into Z_4, and Z_4 is then known when the balance is found.

In the second class Z_2 and Z_3 are of the same kind, usually resistances, and Z_1 is varied to make

$$Z_1Z_4 = Z_2Z_3.$$

If Z_2 and Z_3 are resistances (known, of course), Z_1 and Z_4 are inverse networks of product Z_2Z_3. An example of this is Maxwell's inductance bridge. The method of forming

Fig. 3.



inverse networks is very well known. Impedances in parallel in one network correspond to impedances in series in the inverse network; an inductance L corresponds to a capacity C, and resistances R_1 and R_2 correspond when $\frac{L}{C} = R_1R_2$ = the constant product $= K^2$, say.

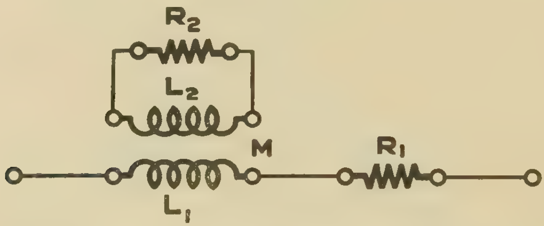
For example, in the simple Maxwell bridge the unknown L and R in series are balanced by the inverse network of C and P in parallel, where

$$\frac{L}{C} = PR = K^2.$$

The equivalence holds at all frequencies, neglecting the changes of capacity, resistance, and inductance with frequency, so that a perfect balance is obtained.

Given the configuration of the unknown impedance, it is simple to see what inverse network corresponds to it for the purpose of obtaining the perfect balance.

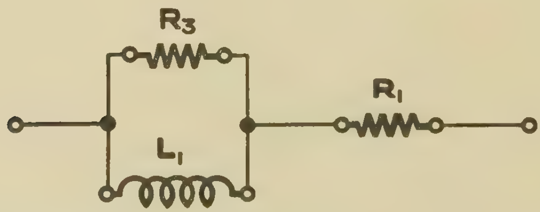
Fig. 4 *a*.



Some Examples of Inverse Networks in Bridges.

As an example, consider the case of the inverse network bridge in which the unknown arm is represented by the configuration shown in fig. 4 *a*. If the coupling is tight so that $M = \sqrt{L_1L_2}$, this may be replaced (using the equivalent T-network of a transformer) by the impedance configuration of fig. 4 *b*, where $R_2 = \frac{L_1}{L_2}R_2$.

Fig. 4 *b*.



The inverse network of this is given in fig. 5, where

$$\frac{L_1}{C} = R_3R_5 = R_1R_4 = k^2.$$

This forms the modified Maxwell bridge shown in fig. 6 in the paper by Harris and Williams quoted above.

$$\frac{L_1}{C} = R_3R_5 = \frac{L_1}{L_2}R_2R_5 \quad \text{gives} \quad \frac{L_2}{R_2} = CR_5,$$

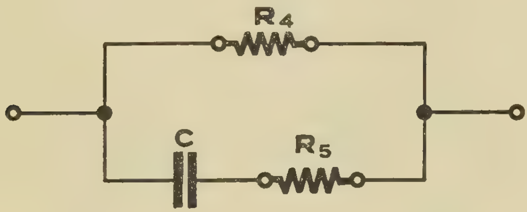
which is the relation found there.

It can be shown by the method given above that the conditions for a ballistic balance are

$$\frac{L_1}{C} = R_1 R_4 = k^2,$$

so that a ballistic balance determines L_1 and R_1 in terms of the variables R_4, C, and k^2, R_5 being immaterial. Then a

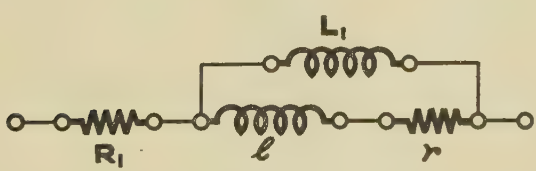
Fig. 5.



current balance determines R_5, and hence $\frac{L_2}{R_2}$, without the necessity of changing C and R_4.

The network of fig. 5 can be replaced by an equivalent network composed of a condenser shunted by a resistance, the whole being in series with another resistance. This is the form of the Rimington-Maxwell bridge, which is not so convenient as the modified Maxwell bridge given above for

Fig. 6.



the case of a coupled secondary. This is shown by Harris and Williams in the work quoted.

When the coupling between the primary and secondary is not perfect, $M \neq \sqrt{L_1 L_2}$. The equivalent network is then shown in fig. 6, where

$$\left.\begin{aligned} l &= L_1(L_1L_2 - M_2)/M^2 \\ \text{and} \qquad r &= R_2 \frac{L_1{}^2}{M^2}. \end{aligned}\right\} \quad \dots\dots \quad (5)$$

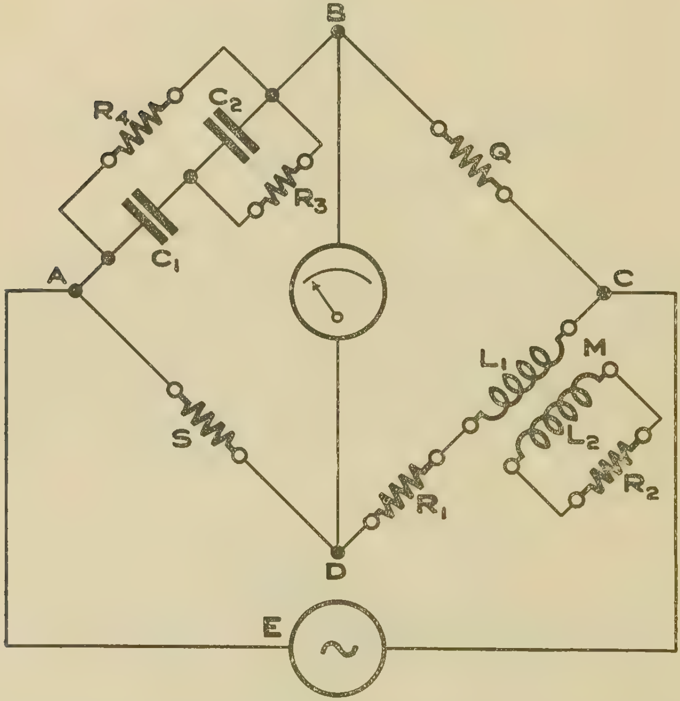
The inverse network is shown in the branch AB of fig. 7, which shows the bridge for a perfect balance.

The conditions for the perfect balance are

$$R_1R_4 = \frac{L_1}{C_1} = \frac{l}{C_2} = rR_3 = QS, \quad . \quad . \quad . \quad (6)$$

l and r being given by the equations (5).

Fig. 7.



The arm AB can be replaced by a variety of equivalent networks, each containing two capacities and two resistances. Closer consideration is needed to see which configuration is the most convenient for practical working. An analogous method of approach can be found, as in the modified Maxwell bridge, in the following way :—The network of branch AB in fig. 7 can be shown to be the most convenient configuration. It is practically impossible to find a bridge balance with four variables, but with the configuration of fig. 7 this can be done in two stages, finding two elements at a time, and

this is quite practicable. First, a ballistic balance can be found by varying only C_1 and R_4. It can be proved quite easily by the general method shown above that then

$$R_1R_4 = \frac{L_1}{C_1} = QS\,(=k^2).$$

This will be proved later by an even simpler method than that given above. Then a current balance is obtained, keeping C_1 and R_4 fixed, and varying C_2 and R_3. From these and C_1 and R_4, previously obtained, it is possible to calculate

$$\frac{L_1L_2-M^2}{M^2} = \alpha, \text{ say, and } R_2\frac{L_1{}^2}{M^2} = \beta, \text{ say.}$$

Then

$$\left.\begin{aligned} M^2 &= L_1L_2/(1+\alpha) \\ \text{and}\quad \frac{R_2}{L_2} &= \frac{\beta}{L(1+\alpha)}. \end{aligned}\right\}$$

The time constant of the secondary is thus obtained.

Measurement of Negative Resistance by the Inverse Network Bridge.

If Z_2 and Z_3 are capacities, corresponding impedances in Z_1 and Z_4 are capacities and positive and negative resistances. This gives a method for measuring negative resistances. The unknown impedance, $-R+jx$, say, is put in the arm Z_4. If x is positive it is neutralized by capacity in series with it, and a positive resistance for Z_1 produces a balance. If x is negative Z_1 is a resistance and capacity in parallel. This holds if inductances are replaced by capacities and capacities by inductances.

A Theorem connecting Equal Ratio and Inverse Network Bridges.

"For a given inverse network bridge that satisfies any criterion of balance there corresponds an equal ratio bridge which satisfies the same criterion, and conversely for a given equal ratio bridge there corresponds an inverse network bridge."

The corresponding bridges are given in figs. 8 *a* and 8 *b*. The proof is as follows:—

The transfer admittance for the bridge of fig. 8 *a* is given by

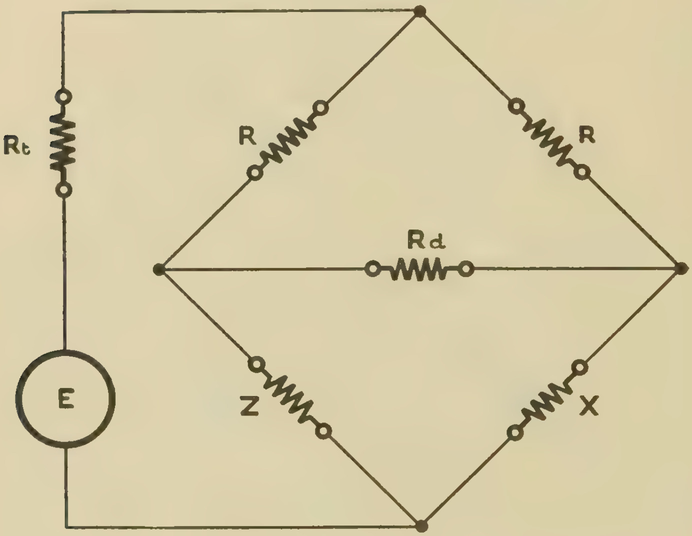
$$f(p) = \frac{R(X-Z)}{A+B(X+Z)+DXZ}, \qquad (7)$$

$$\left.\begin{aligned} \text{where}\qquad A &= R(2R_TR_D + RR_D), \\ B &= R_TR_D + R(R + 2R_T + R_D), \\ \text{and}\qquad D &= 2R + R_D. \end{aligned}\right\}$$

The transfer admittance for the bridge of fig. 8 *b* is given by

$$f(p) = \frac{R^2(X - Z)}{A' + B'X + B''Z + D'XZ}, \quad . \quad . \quad . \quad (8)$$

Fig. 8 *a*.



$$\left.\begin{aligned} \text{where}\qquad A' &= R^2(R^2 + R_TR_D + RR_T + RR_D), \\ B' &= R^2(2R + R_T + R_D), \\ B'' &= R(RR_T + RR_D + 2R_TR_D), \\ \text{and}\qquad D'' &= R_TR_D + R^2 + R(R_T + R_D). \end{aligned}\right\}$$

The admittances of equations (7) and (8) have denominators which are positive polynominals in the operator p of the same power, whilst the numerators are equal except for the constant R. Therefore if one satisfies the conditions for any kind of balance the other will be so too.

As a corollary to this theorem it follows that for any given criterion it is necessary to consider only one of these two types of bridges, say the equal ratio bridge.

Theorems on Ballistic Bridges.

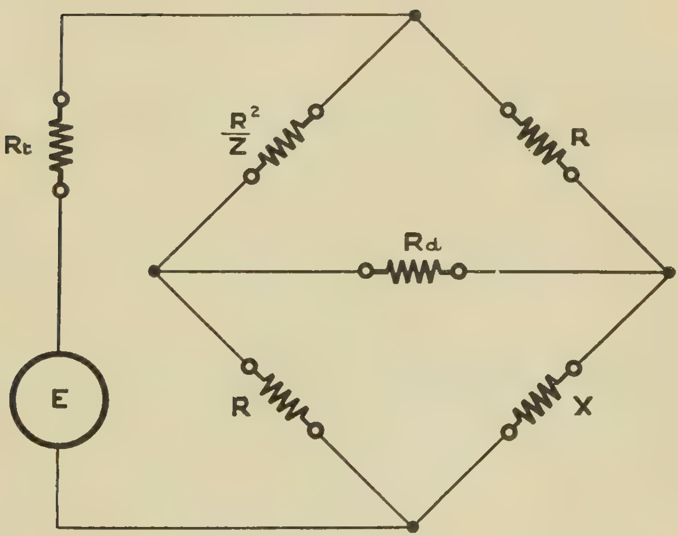
"If the bridge of fig. 8 *a* (or 8 *b*) gives a ballistic balance by the impedance Z for the unknown impedance X, where X can be expanded in powers of p as

$$X = a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \dots,$$

then Z will give a ballistic balance for an impedance X′, where X′ can be expanded as

$$X' = a_0 + a_1 p + a_2' p^2 + a_3' p^3 + \dots"$$

Fig. 8 *b*.



This follows by the conditions of balance as given in equation (4). Here $m=0$. Then

$$Z = a_0 + a_1 p + a_2'' p^2 + a_3'' p^3 + \dots$$

is the necessary condition for a ballistic balance, so that

$$X - Z = (a_2' - a_2'')p^2 + (a_3' - a_3'')p^3 + \dots,$$

the coefficients of p^0 and p' vanishing. The coefficients of p^0 and p' in $(X' - Z)$ vanish also, proving the theorem. The theorem needs further consideration if Z and X contain negative powers of p. This is fully treated in Appendix II.

For a d.c. balance all that is necessary is that the coefficient of p^0 in Z, X, and X′ should be equal. For the ballistic balance one more coefficient, viz., that of p', must be equal in the impedances. This may be expressed by saying that the ballistic balance is the case of balance for a.c. of very low frequency. This is not obvious, as a suddenly applied d.c. wave contains all frequencies from zero to infinity.

A theorem that can be deduced immediately from this is the well-known fact that "the presence of a coupled secondary makes no difference to the ballistic balance for an inductance coil." For let the coil and its secondary be represented by the configuration of fig. 4 *a*, which is equivalent to that of fig. 6: the impedance operator is

$$R_1+\frac{pL_1(r+pl)}{pL_1+r+pl}=R_1+pL_1-p^2\frac{L_1{}^2}{r}+\ldots$$

Since only the coefficients of p^0 and p' have any effect in a ballistic balance, only R_1 and L_1 are determined by such a balance. This proves the validity of the method for finding R_1, L_1, and $\frac{R_2}{L_2}$ by the ballistic and current balances described earlier.

The Possible Cases of a Ballistic Bridge.

We will consider only the bridge of fig. 8 *a*. X is the unknown to be measured and Z is the variable standard. Then it is proved in Appendix II. that there are only three possible cases:

Case 1.—X contains a capacitive part at low frequencies, so that

$$X=\frac{a_{-1}}{p}+a_0+a_1p+\ldots,$$

expanded in ascending powers of p.

The condition of ballistic balance is that Z should contain an equal capacitive part, *i. e.*,

$$Z=\frac{a_{-1}}{p}+a_0{}'+a_1{}'p+\ldots,$$

where $a_0{}'$, $a_1{}'$, ... are any quantities, zero most conveniently.

Case 2.—X contains no capacitive part, but a resistive and inductive part, so that

$$X=a_0+a_1p+a_2p^2+\ldots$$

The conditions of balance are that Z must have resistive and inductive parts equal to those of X, *i. e.*,

$$Z = a_0 + a_1 p + a_2'' p^2 + \ldots,$$

a_2'', a_3'', ... being any quantities.

Case 3.—X contains no capacitive or resistive part, so that

$$X = a_1 p + a_2 p^2 + \ldots$$

(This is generally not the case in practice.) The condition of balance is that Z should have no capacitive or resistive part, but an equal inductive part, *i. e.*,

$$Z = a_1 p + a_2'' p^2 + \ldots,$$

where a_2'' ... are any quantities.

Appendix I.

Proof of the Conditions for Perfect or Ballistic Balance.

When an e.m.f. $E(t)$ is introduced into any part of the bridge the current in the detector is given by

$$i = f(p) E(t), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $f(p)$ is the admittance operator and is calculated by the usual method, putting p instead of $j\omega$.

Equation (1) can be expressed in operational form by

$$\begin{aligned} i &= f(p) \int_{-\infty}^{+\infty} E(h) e^{-ph} p \, . \, dh \, . \, H(t) \\ &= f(p) g(p) H(t) \\ &= \frac{p^{-m}(a_0 + a_1 p + a_2 p^2 + \ldots)}{b_0 + b_1 p + b_2 p^2 + \ldots} g(p) H(t), \quad \text{say}, \quad (2) \end{aligned}$$

where $H(t)$ is Heaviside's unit function given by

$$H(t) = 1, \quad t > 0,$$

and

$$H(t) = 0, \quad t < 0.$$

This can be translated back into ordinary form as

$$i = \frac{1}{2\pi j} \int_{L} e^{pt} f(p) g(p) \, dp/p, \quad . \quad . \quad . \quad . \quad (3)$$

where L is a path from $c-j\infty$ to $c+j\infty$, where c is positive and finite, such that all the singularities of the integrand are on the left side of the path. It is usually unnecessary to have recourse to the general formula (3).

The conditions for a perfect balance are obtained immediately from equation (2). The conditions are that the admittance operator vanishes always, so that

$$a_0 = a_1 = a_2 = \ldots = 0. \quad . \quad . \quad . \quad . \quad (4)$$

The conditions for a ballistic balance are obtained as follows :—

The e.m.f. $\mathrm{E}(t)$ is taken as $\mathrm{H}(t)$, so that

$$\begin{aligned} i &= f(p)\mathrm{H}(t) \\ &= \frac{p^{-m}(a_0+a_1p+a_2p^2+\ldots)}{b_0+b_1p+b_2p^2+\ldots}\mathrm{H}(t). \end{aligned}$$

The integrated current up till time t is given by

$$\begin{aligned} \mathrm{Q} &= \int^t i\,dt = \frac{1}{p}i = \frac{1}{p}f(p)\mathrm{H}(t) \\ &= \frac{p^{-m-1}(a_0+a_1p+a_2p^2+\ldots)}{b_0+b_1p+b_2p^2+\ldots}\mathrm{H}(t) \quad . \quad . \quad (5) \\ &= \frac{\phi(p)}{\Delta(p)}\mathrm{H}(t). \end{aligned}$$

In all physical systems $b_0, b_1, b_2, \ldots$ are positive. Expression (5) can be expanded by the Heaviside expansion theorem as

$$\frac{\phi(0)}{\Delta(0)} + \sum_r \frac{\phi(p_r)}{p_r\Delta'(p_r)}e^{p_rt}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where p_r are the roots of

$$b_0+b_1p+b_2p^2+\ldots = 0,$$

and are therefore all negative. As t tends to ∞,

$$\begin{aligned} \mathrm{Q} \text{ tends to } \mathrm{Q}\infty &= \frac{\phi(0)}{\Delta(0)} \\ &= \left[p^{-m-1}(a_0+a_1p+\ldots)/b_0\right]_{p=0} \\ &= \left[\frac{f(p)}{p}\right]_{p=0}. \end{aligned}$$

The condition for a ballistic balance is thus obtained as

$$\left[\frac{1}{p}f(p)\right]_{p=0} = 0, \quad \ldots \ldots \quad (7)$$

where $f(p)$ is the admittance operator for the detector.

Expressing $f(p)$ as the ratio of two polynominals, the conditions become

$$a_0 = a_1 = \ldots = a_{m+1} = 0, \quad \ldots \quad (8)$$

which are the conditions quoted.

Usually $m=0$, so that the conditions are

$$a_0 = a_1 = 0. \quad \ldots \ldots \quad (9)$$

It has been shown that if

$$\mathrm{F}(p)\mathrm{H}(t) = \psi(t),$$

then

$$\lim_{t \to 0} \psi(t) = \mathrm{F}(0).$$

It can be shown by a similar use of the Heaviside expansion theorem that

$$\lim_{t \to 0} \psi(t) = \mathrm{F}(\infty).$$

Appendix II.

Ballistic Balance when the Impedances may contain a Negative Power of p.

The expansion of an impedance operator in ascending powers of p is always expressible in the form

$$\mathrm{X} = \frac{a_{-1}}{p} + a_0 + a_1 p + a_2 p^2 + \ldots, \quad \ldots \quad (1)$$

where a_{-1}, a_0, and a_1 cannot simultaneously vanish. Let the impedance Z in the bridge of figs. 8 *a* or 8 *b* be a similar expression with *b*'s in place of the *a*'s.

Suppose a_{-1} does not vanish, for the case when it does has been treated before. Then the admittance operators of equations (7) and (8) become of the form

$$f(p) = p^2 \left[\frac{\frac{1}{p}(a_{-1}-b_{-1}) + (a_0-b_0) + p(a_1-b_1) + \ldots}{\mathrm{D} + \mathrm{E}p + \mathrm{F}p^2 + \ldots}\right]$$

$$= \frac{p(a_{-1}-b_{-1}) + p^2(a_0-b_0) + \ldots}{\mathrm{D} + \mathrm{E}p + \mathrm{F}p^2 + \ldots}, \quad \ldots \ldots \quad (2)$$

where D, E, and F are positive constants for the case when b_{-1} does not vanish. Condition (8) of Appendix I. give as the ballistic balance requirements merely that

$$a_{-1} = b_{-1}, \quad \ldots \ldots \ldots \quad (3)$$

i. e., the capacitive parts of Z and X must be equal.

If b_{-1} does vanish, $f(p)$ becomes

$$f(p) = p\frac{\frac{a_{-1}}{p} + (a_0 - b_0) + p(a_1 - b_1)}{\mathrm{D} + \mathrm{E}p + \mathrm{F}p^2 + \ldots}$$

$$= \frac{a_{-1} + p(a_0 - b_0) + p^2(a_1 - b_1)}{\mathrm{D} + \mathrm{E}p + \mathrm{F}p^2 + \ldots} \quad \ldots \quad (4)$$

For a ballistic balance

$$\left.\begin{aligned} a_{-1} &= 0 \\ \text{and} \qquad a_0 &= b_0, \end{aligned}\right\}$$

so that a ballistic balance is impossible unless Z has a capacitive part $\frac{a_{-1}}{p}$ to correspond with that of Z. There are thus only three cases in ballistic bridges :—

I. If X contains a term $\frac{a_{-1}}{p}$, Z must contain an equal term, and this is the only condition of balance.
  The m of equation (2), Appendix I. is equal to -1, and there is only one condition to be satisfied.

II. If $a_{-1} = 0$ but $a_0 \neq 0$, it is easily seen that $m = 0$, and there are two conditions of balance, viz., $a_0 = b_0$ and $a_1 = b_1$.

III. If $a_{-1} = a_0 = 0$, then a_1 cannot vanish. Then $m = -1$, and the condition of ballistic balance is $a_1 = b_1$.

XXI. *An Integral involving the Elliptic Cylinder Function.* *By* Rama Shankar Varma, *M.Sc.* *

IN view of the increasing importance of definite integrals in physical problems, as well as from the purely mathematical aspect, it seems desirable to place on record the value of the integral

$$\int_0^x y(\alpha, q) \cos 2\alpha \sin \sqrt{\mathrm{A}}(\alpha - x) d\alpha,$$

* Communicated by the Author.

where $y(x, q)$ is a solution, even or odd, of the first or second kind, of Mathieu's equation,

$$\frac{d^2y}{dx^2}+(\mathrm{A}+16q\cos 2x)y=0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

for a certain relation between A and q.

For this purpose we write (1) in the form

$$y''(\alpha, q)+(\mathrm{A}+16q\cos 2\alpha)y(\alpha, q)=0, \quad . \quad . \quad (2)$$

where dashes denote differentiations.

If $\mathrm{S_A}=\sin(\sqrt{\mathrm{A}}\alpha)$ and $\mathrm{C_A}=\cos(\sqrt{\mathrm{A}}\,\alpha)$, we get

$$\mathrm{S_A}''+\mathrm{A\,S_A}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and

$$\mathrm{C_A}''+\mathrm{A\,C_A}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Multiply (2) by $\mathrm{S_A}(\alpha)$ and (3) by $y(\alpha, q)$, and subtract. We obtain

$$\begin{aligned} y''(\alpha, q)\mathrm{S_A}(\alpha)-\mathrm{S_A}''(\alpha)y(\alpha, q)& \\ =-16q\,\mathrm{S_A}(\alpha)\cos 2\alpha\, y(\alpha, q).& \quad . \quad (5)\end{aligned}$$

Similarly from (2) and (4) we get

$$\begin{aligned} y''(\alpha, q)\mathrm{C_A}(\alpha)-\mathrm{C_A}''(\alpha)y(\alpha, q)& \\ =-16q\,\mathrm{C_A}(\alpha)\cos 2\alpha\, y(\alpha, q).& \quad . \quad (6)\end{aligned}$$

From (5) and (6) it follows that

$$\begin{aligned} \Big| y'(\alpha, q)\mathrm{S_A}(\alpha)-\mathrm{S_A}'(\alpha)y(\alpha, q) \Big|_0^x& \\ =-16q\int_0^x \mathrm{S_A}(\alpha)\cos 2\alpha\, y(\alpha, q)\,d\alpha,& \quad . \quad (7)\end{aligned}$$

and

$$\begin{aligned} \Big| y'(\alpha, q)\mathrm{C_A}(\alpha)-\mathrm{C_A}'(\alpha)y(\alpha, q) \Big|_0^x& \\ =-16q\int_0^x \mathrm{C_A}(\alpha)\cos 2\alpha\, y(\alpha, q)\,d\alpha.& \quad . \quad (8)\end{aligned}$$

Evidently

$$\mathrm{S_A}(0)=0, \qquad \mathrm{S_A}'(0)=\sqrt{\mathrm{A}},$$
$$\mathrm{C_A}(0)=1, \qquad \mathrm{C_A}'(0)=0\,;$$

hence (7) and (8) can be written as

$$\begin{aligned} y'(x, q)\mathrm{S_A}(x)-y(x, q)\mathrm{S_A}'(x)+\sqrt{\mathrm{A}}\,y(0, q)& \\ =-16q\int_0^x \mathrm{S_A}(\alpha)\cos 2\alpha\, y(\alpha, q)\,d\alpha,& \quad . \quad . \quad . \quad (9)\end{aligned}$$

and

$$y'(x, q)C_A(x) - y'(0, q) - y(x, q)C_A'(x)$$

$$= -16q\int_0^x C_A(\alpha)\cos 2\alpha\, y(\alpha, q)\,d\alpha. \quad . \quad . \quad . \quad (10)$$

Multiply (9) by $S_A'(x)$ and (10) by $C_A'(x)$ and add. We get, since

$$S_A S_A' + C_A C_A' = 0 \quad \text{and} \quad S_A'^2 + C_A'^2 = A,$$

$$\sqrt{A}\, y(x, q) - \sqrt{A}\cos(\sqrt{A}\, x)y(0, q) - \sin(\sqrt{A}\, x)y'(0, q)$$

$$= 16q\int_0^x y(\alpha, q)\cos 2\alpha \sin\sqrt{A}(\alpha - x)\,d\alpha.$$

Interesting particular cases of this general integral are given by putting $x = \frac{\pi}{2\sqrt{A}}$ and $x = \frac{\pi}{\sqrt{A}}$; we find then that

$$y'(0, q) - \sqrt{A}\, y\left(\frac{\pi}{2\sqrt{A}}, q\right)$$

$$= 16q\int_0^{\frac{\pi}{2\sqrt{A}}} y(\alpha, q)\cos 2\alpha \cos\sqrt{A}\,\alpha\,d\alpha, \quad . \quad (11)$$

and

$$-\sqrt{A}\, y(0, q) - \sqrt{A}\, y\left(\frac{\pi}{\sqrt{A}}, q\right)$$

$$= 16q\int_0^{\frac{\pi}{\sqrt{A}}} y(\alpha, q)\cos 2\alpha \sin\sqrt{A}\,\alpha\,d\alpha. \quad (12)$$

If $E(x, q)$ be even Mathieu functions, we deduce easily from (11) that

$$\int_0^\pi E\left(\frac{\epsilon}{2\sqrt{A}}, q\right)\cos\frac{\epsilon}{\sqrt{A}}\cos\frac{\epsilon}{2}\,d\epsilon = -\frac{A}{8q}E\left(\frac{\pi}{2\sqrt{A}}, q\right).$$

Similarly from (12)

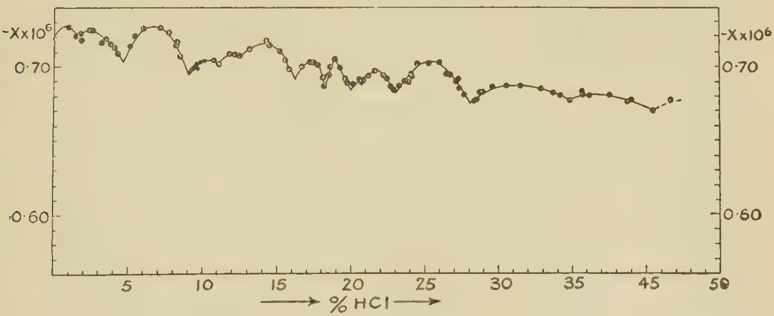
$$\int_0^\pi 0\left(\frac{\epsilon}{\sqrt{A}}, q\right)\cos\frac{2\epsilon}{\sqrt{A}}\sin\epsilon\,d\epsilon = -\frac{A}{16q}0\left(\frac{\pi}{\sqrt{A}}, q\right),$$

where $0(x, q)$ is an odd Mathieu function.

XXII. *The Magnetism of Binary Mixtures: Aqueous Solutions of Acids.* *By* JOHN FARQUHARSON, *B.Sc.*, *Ph.D.*, *Senior* 1851 *Student* *.

IT is of interest to examine the susceptibilities of binary mixtures of diamagnetic liquids and to observe the change of susceptibility with concentration. It is well known that magnetic measurements give a convenient method for indicating the presence of compound formation between two substances. Much work has been done from time to time on aqueous solutions in order to determine experimental values for free ions. Especially is this true for HCl, where, were the property strictly additive and the ions free in

Fig. 1.



Susceptibility-concentration curve for HCl.

solution, one would have a convenient method for determining the diamagnetism of the free Cl^- ion.

Hocart (*Compt. Rend.* clxxxviii. p. 1151, 1929) studied solutions of HCl at seven different concentrations, and the susceptibilities of these solutions lie approximately on a straight line and give a mean value for the molecular susceptibility of HCl as $-22{\cdot}0 \times 10^{-6}$. A comparison of his few results with the susceptibility-concentration curve for HCl, given in fig. 1, shows an almost quantitative agreement. Hocart states that he found a small deviation from additivity.

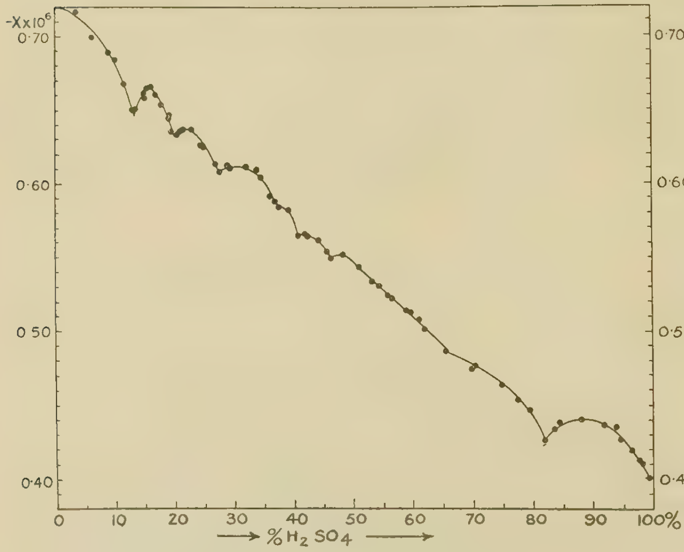
It is apparent that one must study the diamagnetism at a very large number of concentrations, and if this is done then it is found that there is a large departure from additivity, as shown by fig. 1 for HCl and by fig. 2 for H_2SO_4.

* Communicated by Prof. F. G. Donnan, Sc.D., F.R.S.

U 2

The value for pure HCl is unknown, and it is thus impossible to find the actual deviation from additivity; but there is no reason to suppose that the mean value for HCl calculated from measurements of solutions is the correct one when the departure from additivity exhibited by H_2SO_4 is considered. It is probable that the almost linear part of the curve, extending from about 30 per cent. HCl upwards, corresponds to the linear part of the H_2SO_4 curve extending from 50 per cent. to 80 per cent. H_2SO_4. If this is so, then the value for pure HCl will be larger than that calculated from measurements in solution.

Fig. 2.



Susceptibility-concentration curve for H_2SO_4.

Weiss (*J. de Phys. et Rad.* vii. (6) p. 185, 1930) has discussed the results of Hocart and of others on HCl, the alkali halides, and the alkaline earth halides in an analogous way to that used by Joos and Fajans when discussing refraction and the deformation of ions. He indicates that diamagnetism is a half less sensitive to this deformation than is refraction. The H^+ ion in solution diminishes the refraction of water to a sensible extent, whereas the halogen ions, in virtue of their negative charge, tend to increase the refraction; but the effect is negligible for Cl^-. Weiss

deduces that the loss of diamagnetism or the apparent paramagnetism of H^- is $10^6 X_{H^+ diss.} = 1 \cdot 1$.

In discussing the experimental results consideration must be given to ionization, to hydration, and to the lowering of the susceptibility of water by the H^+ ion. This last may be accounted for by the formation of the hydronium ion H_3O^+.

It is fairly certain, from the large amount of work done on solvation, that the Cl^- ion is hydrated in HCl solution and that HCl is completely dissociated into its ions. The susceptibility measurements are in agreement with this. If we examine the HCl curve we find well-defined maxima and minima on the curve, and passing from higher to lower

TABLE I.

Concentrations of HCl at which Hydrates are calculated and are observed.

Hydrates. Mols. H_2O.	Calculated. Per cent. HCl.	Observed. Per cent. HCl.
1	67·00	—
2	50·35	—
3	40·30	40·5
4	33·65	35·0
5	28·36	28·4
6	25·25	25·4
7	22·46	23·0
8	20·22	20·0
9	18·40	18·2
10	16·86	16·4

concentrations on the curve we find a wonderful agreement between the observed minima and the concentration at which we expect to find hydrates.

Complete agreement is not to be expected, because of the formation of equilibrium mixtures.

After the minimum at 16·4 per cent. the curve changes its nature, and the remaining maxima and minima are much more widely spaced. It is suggested, therefore, that the measurement of susceptibility is a good method of ascertaining the degree of hydration of an ion, and that the maximum hydration of the Cl^- ion is $10H_2O$. The 6-hydrate shows a maximum, which indicates that here the water is bound in a way different from the others. The explanation of this may be found in the present-day knowledge of atomic structure ; Cl^- has a completed subgroup of six electrons.

We know that the perfectly free Cl^- ion would have a susceptibility much larger than in its compounds and in solution, and it is to be assumed that the highest point on the susceptibility-concentration curve would represent the Cl^- ion in its freest state. This point is taken to be the maximum at 6·6 per cent. HCl with a gram susceptibility of $-{\cdot}727\times10^{-6}$. (The maximum at the low concentration of 1 per cent. HCl is not taken, owing to uncertainty of measurement.) Now, if we calculate the gram ionic susceptibility, using this value of $\cdot727\times10^{-6}$, we get a value for the Cl^- ion of $-25{\cdot}8\times10^{-6}$. Again, as Weiss has pointed out, there is a diminution in the susceptibility of water by the H^+ ion, and if we allow a diminution of 1·1 for each of a number of water molecules equivalent to the number of H^+ ions present and recalculate the gram ionic susceptibility of Cl^-, we get a value of $-30{\cdot}66\times10^{-6}$.

Pauling (Proc. Roy. Soc. A, cxiv. p. 181, 1927) has calculated on theoretical grounds the susceptibilities of free atoms and ions, and gives for Cl^- a value of $-29{\cdot}00\times10^{-6}$. Stoner (Proc. Leeds Phil. Soc. (1) x. p. 484, 1929), using Hartree's self-consistent field method, has calculated the ionic susceptibilities of a number of free ions, and gives a value for Cl^- of $-40{\cdot}39\times10^{-6}$; but Brindley and Wood (Phil. Mag. vii. p. 616, 1929) say that this is subject to a reduction, and give a value of $-36{\cdot}7\times10^{-6}$.

The value of $-30{\cdot}66\times10^{-6}$, deduced in this investigation, is the value for the Cl^- ion in its most diffuse state in HCl solution, and it is probable that the Cl^- ion in solution does have a value higher than $-19{\cdot}5$ (Joos), $-20{\cdot}4$ (Ikenmeyer), -22 (Hocart), and $-23{\cdot}1$ (Weiss). If this is so, then the discrepancy between the experimental and theoretical values for K^+ and Na^+ will be explained (*cf.* Stoner, 'Magnetism,' Methuen, 1930, pp. 22–26). The experimental values for these positive ions are not compatible with the values for the rare gases found by Wills and Hector. Using a higher value for Cl^- would lower the values of K^+ and Na^+ and give a better agreement with theory.

The curve for sulphuric acid shows a flat maximum at 86–89 per cent., indicating the presence of the first hydrate of H_2SO_4 (87·49 per cent.). The curve is almost linear except for a very slight indication of a break at 65 per cent. ($H_2SO_4.3H_2O=64{\cdot}3$ per cent.). The curve shows only one decided maximum at 15·1 per cent. H_2SO_4, which is an additive value. If the ions are in their freest state at this point, the value for the $SO_4^{=}$ ion is $-39{\cdot}0\times10^{-6}$. One of the other maxima probably represents the HSO_4^- ion

Experimental.

The measurements were made on an improved Curie-Cheneveau magnetic balance, magnetically damped with rotating permanent magnet, and a quartz suspension. The improvements consisted of copper screening, alterations to the carrier of the damping magnet to allow of horizontal and vertical movement of the magnet to give easier manipulation, and attaching rigidly the quartz fibre to the torsion head and to the beam of the balance. The whole balance was placed in an electrically heated and controlled air thermostat at 18° C., and measurements were made by means of a galvanometer lamp and scale, a glass window being fitted in the copper screen, and in the thermostat a small shutter, which could be opened when a measurement was in progress. The maker's twin-rein method of rotating the magnet was discarded for a micrometric method. These alterations counteract to a marked extent the difficulties produced by outside electrical and magnetic disturbances. Measurements have an accuracy of at least 0·50 per cent.

The solutions were compared with a standard substance—in this investigation conductivity water—of which the susceptibility was taken to be $-0{\cdot}72 \times 10^{-6}$.

Hydrochloric Acid.—The hydrogen chloride was prepared from chemically pure hydrochloric acid as supplied to the laboratory. The gas was driven off either by means of the dropping method with concentrated H_2SO_4 or by heating. First of all it was absorbed in ice-cold distilled water until this was saturated, and then this was heated and the gas absorbed in ice-cold conductivity water. The solutions so prepared were stored in conical flasks with ground stoppers. Resistance glass was used throughout, as soft glass gave rise to impurity and anomalous results. Less concentrated solutions were obtained by diluting with conductivity water. All solutions were allowed to age from two to three days before measurement.

After each measurement the quantity of HCl solution in the small measuring tube was washed carefully into a vessel and titrated with $\frac{N}{10}$ NaOH; thus strength of the acid measured was known exactly. At first it was titrated also with $\frac{N}{10}$ $AgNO_3$, but the two results were so nearly equal that this practice was discontinued.

Sulphuric Acid.—Sulphuric acid solutions of concentrations up to 98 per cent. were prepared by diluting the 98 per cent.

Table II.

Experimental Values of the Susceptibility of HCl Solutions used in the Graph.

Per cent. HCl.	$-X \times 10^6$.	Per cent. HCl.	$-X \times 10^6$.	Per cent. HCl.	$-X \times 10^6$.
42·00	·6790	22·80	·6860	13·18	·7121
40·53	·6703	22·49	·6921	12·53	·7065
39·05	·6776	22·27	·6940	12·20	·7085
38·90	·6760	21·73	·6970	11·82	·7090
37·73	·6801	21·71	·6980	11·05	·7011
36·30	·6795	21·29	·6937	10·86	·7005
35·99	·6802	20·91	·6900	10·27	·7043
35·88	·6831	20·86	·6890	10·13	·6981
35·01	·6768	20·62	·6920	9·85	·7040
34·30	·6804	20·25	·6885	9·76	·7070
33·94	·6770	19·78	·6894	9·61	·7031
33·83	·6820	19·75	·6917	9·60	·6910
33·00	·6847	19·34	·6990	9·49	·7012
31·62	·6870	18·96	·7050	9·35	·6980
30·72	·6873	18·65	·6937	9·14	·6970
29·00	·6833	18·56	·6937	8·43	·7067
28·74	·6830	18·48	·6786	8·20	·7176
28·65	·6770	18·45	·6903	8·20	·7145
28·61	·6770	18·18	·6864	7·72	·7235
28·42	·6766	18·05	·6923	7·11	·7260
27·79	·6808	17·81	·7018	6·04	·7258
27·41	·6857	17·79	·6998	5·40	·7210
27·40	·6918	17·49	·7031	5·31	·7211
27·18	·6900	17·34	·7041	5·11	·7135
26·78	·6956	17·28	·6980	4·35	·7092
26·55	·6949	17·10	·6970	4·08	·7130
26·15	·7037	17·02	·6960	3·91	·7149
25·38	·7021	16·75	·7000	3·47	·7166
24·58	·7024	16·25	·7070	2·58	·7250
24·12	·6936	15·84	·6981	2·45	·7248
24·08	·6954	15·65	·7035	1·90	·7180
24·00	·6889	15·20	·7115	1·80	·7224
23·28	·6877	14·53	·7140	1·54	·7214
22·96	·6840	14·29	·7180	1·01	·7277

acid obtained by the distillation of concentrated chemically pure sulphuric acid as supplied to the laboratory. The distillation was carried out in an apparatus made wholly of resistance glass, the receiver being connected by means of a

Table III.

Experimental Values of the Susceptibility of H_2SO_4 Solutions used in the Graph.

Per cent. H_2SO_4.	$-X\times10^6$.	Per cent. H_2SO_4.	$-X\times10^6$.	Per cent. H_2SO_4.	$-X\times10^6$.
99·45	·4022	55·77	·5254	27·49	·6095
98·18	·4120	54·49	·5318	26.97	·6141
98·01	·4142	53·38	·5350	24·84	·6263
97·60	·4160	51·35	·5457	24·21	·6273
96·66	·4204	51·14	·5441	22·67	·6378
95·06	·4321	48·36	·5535	21·34	·6394
94·03	·4333	46·31	·5502	20·79	·6348
92·25	·4397	45·60	·5520	19·51	·6369
88·42	·4438	45·60	·5548	19·48	6365
84·60	·4399	44·36	·5622	18 89	·6483
84·18	·4384	44·24	·5641	18·80	·6451
83·98	·4467	44·07	·5618	17·49	·6550
82·34	·4362	42·44	·5649	16·72	·6613
79·68	·4476	42·30	·5678	15·99	·6670
77·58	·4540	42·04	·5672	15·16	·6660
75·03	·4642	40·84	·5645	14·98	·6587
70·58	·4784	39·06	·5830	14·59	·6630
70·23	·4751	37·40	·5857	13·44	·6522
65·60	·6872	36·84	·5894	12·83	·6514
62·13	·5015	35·95	·5921	11·45	·6692
61·98	·5038	34·48	·6049	9·88	·6860
61·21	·5086	34·01	·6103	8·71	·6907
59·69	·5128	31·94	·6122	6·40	·7010
59·05	·5153	29·11	·6107	3·19	·7185
56·56	·5233	28·75	·6132		

ground-joint and fitted with a guard-tube. Pure sulphuric acid and sulphuric acid of concentrations higher than 98 per cent. were prepared by adding SO_3 to the 98 per cent. and then by fractional crystallization.

The highest concentration recorded is 99·45 per cent., but this was after measurement, when the small quantity

measured, less than ·5 c.c., had been transferred by means of a pipette from the storage flask to the measuring tube, and had remained therein for approximately a quarter of an hour, so that in all probability it had absorbed a certain amount of moisture from the atmosphere.

The solutions measured were estimated with standard alkali, as in the case of HCl.

Summary.

1. Curves are found for the change of susceptibility with concentration of HCl and H_2SO_4 in water.

2. The HCl curve shows the presence of the hydrated Cl^- ion in its stages of hydration.

3. Accepting the highest point on the curve as the state in which the Cl^- is most free, a value is computed for the gram ionic susceptibility of Cl^- equal to $-30{\cdot}66\times 10^{-6}$.

In conclusion, I wish to express my thanks to Professor Donnan, C.B.E., F.R.S., for his encouragement and advice throughout this investigation.

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XXIII. *Reaction Rate in the System Solid-Solid-Gas.*

By R. S. Bradley, *M.A.**

WHEN one solid, such as a salt hydrate, decomposes to give another solid and gas the conditions are very complex. In such a reaction as

$$\underset{\text{solid}}{S_1} \rightleftharpoons \underset{\text{solid}}{S_2} + \underset{\text{gas}}{G}$$

reaction starts at the surface of S_1, probably at an edge, because there the forces acting on G are most asymmetric. For the same reason a reaction wave spreads out from this point on the surface of S_1, the reaction rate at this S_1-S_2 interface being greater for macroscopic crystals than that at the interface S_1-G. The fact that the march of the interface S_1-S_2 across the solid is independent of the amount decomposed shows that diffusion through the layer of S_2 to the surface is faster than the reaction rate at the interface.

* Communicated by the Author.

Presumably the reverse reaction can also be neglected if the external pressure of G is small.

The reaction rate may be related to the oscillation frequency of G at the interface S_1-S_2. Frenkel[1] took the phase volume of a molecule oscillating on a surface of area S to be

$$\int \mathrm{S}e^{\frac{u_0-\Delta u}{k\mathrm{T}}}.dz,$$

where u_0 is the energy in the equilibrium position and z is measured perpendicular to the surface. Δu is the potential energy of the simple harmonic motion, and equals

$$2\frac{\pi^2}{\tau_0^2}.mz^2,$$

where τ_0 is the period of the oscillation, m the mass of the molecule G. Hence the phase volume is

$$\int_{-\infty}^{\infty} \mathrm{S}e^{\frac{u_0}{k\mathrm{T}}}.e^{-\frac{2\pi^2mz^2}{\tau_0^2k\mathrm{T}}}.dz = \mathrm{S}e^{\frac{u_0}{k\mathrm{T}}}.\left(\frac{k\mathrm{T}}{2\pi m}\right)^{\frac{1}{2}}.$$

If there are n molecules of G per area S of the interface and n' per volume V of the gas phase (or per volume V of S_2 when the interface is between S_1 and S_2) at the establishment of equilibrium between the two phases, then

$$\frac{n}{\mathrm{S}}=\frac{n'}{\mathrm{V}}\tau_0\left(\frac{k\mathrm{T}}{2\pi m}\right)^{\frac{1}{2}}.e^{\frac{u_0}{k\mathrm{T}}}.$$

The number of gas molecules hitting an area S in unit time is

$$\nu=\mathrm{S}\frac{n'}{\mathrm{V}}\left(\frac{k\mathrm{T}}{2\pi m}\right)^{\frac{1}{2}}.$$

On writing the equilibrium condition $\nu=\frac{n}{\tau}$, where τ is the mean life of a surface molecule, and combining the above results, we get

$$\tau=\tau_0 e^{\frac{u_0}{k\mathrm{T}}}.$$

The rate at which molecules leave the interface, in molecules per sec. per sq. cm., is therefore

$$\frac{n}{\mathrm{S}\tau}=\frac{n}{\mathrm{S}\tau_0}.e^{-\frac{u_0}{k\mathrm{T}}}.$$

This derivation can be criticized. In the first place, it is assumed that the whole surface S is available for the condensation of gas molecules according to $\mathrm{G}+\mathrm{S}_2\rightarrow\mathrm{S}_1$. Then it is

assumed that at every collision with the interface a molecule is held. Finally, the phase volume for a linear oscillator incapable of motion in the surface should include a term for the kinetic energy, giving the value

$$\int_{-\infty}^{\infty} e^{-\frac{m\xi^2}{2kT}} \cdot m\, d\xi \int_{-\infty}^{\infty} e^{-\frac{2\pi^2 m z^2}{\tau_0^2 kT}} dz = (2\pi mkT)^{\frac{1}{2}} \left(\frac{kT}{2\pi m}\right)^{\frac{1}{2}} \cdot \tau_0,$$

where $\frac{1}{2}m\xi^2$ is the kinetic energy. This corresponds to the value $(2\pi mkT)^{\frac{3}{2}}V$ for the gas. Then, also, the fact that one would not expect the surface molecule to have the same number of degrees of freedom of rotation as the molecule in the gas should be included in the above statistical treatment.

For these reasons a more direct treatment is preferable, especially as the direct and reverse rates are quite independent. Suppose, as before, that we have n oscillators per area S of the interface. A molecule can leave the surface only on the swing of an oscillation. If all oscillations were effective $\frac{n}{\tau_0}$ molecules would leave the surface S in unit time.

However, oscillations are only effective for which kinetic energy $\frac{1}{2}m\xi^2$ exceeds the potential energy $u_0 - \frac{2\pi^2 m \cdot z^2}{\tau_0^2}$. Hence the potential and kinetic energy must together be greater than u_0. The chance that the energy Q should lie between Q and $Q+dQ$ in one term of the energy (*e.g.*, kinetic or potential) is [(2)]

$$\frac{Q^{-\frac{1}{2}}}{(\pi kT)^{\frac{1}{2}}} \cdot e^{-Q/kT}\, dQ.$$

The chance that the total energy is between E and $E+dE$ for two terms, without reference to the way in which it is divided between them, is therefore

$$\int_0^E \frac{(Q_1 \cdot Q_2)^{-\frac{1}{2}}}{\pi kT} e^{\frac{-Q_1-Q_2}{kT}} \cdot dQ_1 \cdot dQ_2$$

when Q_2 is put equal to $E-Q_1$, and the integral is evaluated from $Q_1=0$ to $Q_1=E$. The result is $\frac{e^{-E/kT} \cdot dE}{kT}$. Hence the fraction of molecules with energies in the two terms greater than E is

$$\frac{1}{kT}\int_E^{\infty} e^{-E/kT} \cdot dE = e^{-E/kT}.$$

The rate of the solid reaction is therefore

$$\frac{n}{S\tau_0}e^{-u_0/kT} \text{ per sq. cm. per sec., as before.}$$

In the first place this is of the correct form. It resembles Dushman's equation for homogeneous gas reactions, except that Dushman writes $u_0 = \frac{h}{\tau_0}$. Experimental verification of the expression for the mean life of gas molecules on solids has been given by Clausing[(3)], while Cockroft[(4)] has shown that the critical stream densities for the condensation of metallic atoms on surfaces are in accordance with Frenkel's theory, developed from this expression.

It is unfortunate that in the majority of solid reactions the interfacial area has not been measured. In some cases the linear march of the interface has been observed[(5)]. Garner and Tanner[(6)], however, have referred reaction rates for the decomposition

$$CuSO_4.5H_2O \rightarrow CuSO_4.H_2O + 4H_2O$$

to unit interfacial area. The four water molecules which come off are probably attached to the Cu^{++} ion, as the radius of this is so much smaller than that of SO_4^{--}. This is in agreement with the coordination chemistry of copper and with volume considerations similar to those used by Goldschmidt to determine the number of ions which may be grouped about a central ion in a coordination lattice. It is noteworthy that the energy change per gm. mol for the reaction

$$CuSO_4.5H_2O \rightarrow CuSO_4.3H_2O + 2H_2O \text{ (gas)}$$

is 13,268[(7)] cals., and that for the reaction

$$CuSO_4.3H_2O \rightarrow CuSO_4.H_2O + 2H_2O \text{ (gas)}$$

is 13,256 cals.; there can therefore be little interaction between the four dipoles, or at least this is small compared with that between the dipoles and the central ion. We may roughly take the four molecules to be equivalent, even when some of the four have left the cupric ion. We may suppose, also, that as one plane in the crystal becomes denuded the next begins to decompose, so that $\frac{n}{S}$ in the above analysis remains approximately constant.

Garner and Tanner found for the energy of activation the value 10,300 cals., which is practically the same as the latent heat of evaporation per gm. mol. The energy of activation for the reverse reaction, the hydration of the monohydrate, is not likely to be appreciable. The difference between Garner and Tanner's value and 13,000 cals. may therefore be ascribed to the presence of the pseudomorph. In other words, less energy is required to take a molecule G from S_1 to S_2 than from S_1 to the gas phase direct. This corresponds to a greater reaction rate at the interface S_1-S_2 than at the interface S_1—gas: the solid interfacial is greater than the nucleation rate.

From Garner and Tanner's data the reaction rate in gm. of water per sq. cm. per minute is given by

$$1{\cdot}07 \,.\, 10^8 \,.\, e^{-u_0/kT}.$$

This should be equal to

$$\frac{60 \,.\, 18\, ne^{-n_0/kT}}{6{\cdot}06 \,.\, 10^{23} \,.\, S\tau}.$$

For $\frac{n}{S}$ we write

$$4\left(\frac{6{\cdot}06 \,.\, 10^{23} \,.\, \rho}{M}\right)^{\frac{2}{3}},$$

where M is the molecular weight of $CuSO_45H_2O$ and ρ is its density. The factor 4 allows roughly for the presence of four molecules of reacting H_2O per Cu^{++}. On this basis $\frac{1}{\tau_0}$ should be $5 \,.\, 10^8$.

This value is, however, much too small. Consider a dipole of strength μ oscillating about an ion of charge e. The polarizability of the dipole is α, and r is the distance from the ionic to the dipole centre. Then, neglecting interatomic oscillations in the attached dipole,

$$m\frac{d^2r}{dt^2}+\frac{2\mu e}{r^3}+\frac{2\alpha e^2}{r^5}-\frac{\beta}{r^n}=0.$$

The last term represents the repulsive force. The ion is supposed to be rigid. In this equation r can be replaced by r_0+w, where r_0 is the equilibrium distance and w is the displacement; r_0 is then defined by

$$\left(\frac{du}{dr}\right)_{r_0}=0,$$

where u, the energy, is given by

$$u = -\frac{e\mu}{r^2} - \frac{1}{2}\frac{\alpha e^2}{r^4} + \frac{\beta}{(n-1)r^{n-1}} + E.$$

Here E is included to allow for the neighbourhood of the pseudomorph.

The equilibrium condition

$$\left(\frac{du}{dr}\right)_{r_0} = 0$$

gives

$$\frac{2\mu e}{r_0^{\,3}} + \frac{2\alpha e^2}{r_0^{\,5}} - \frac{\beta}{r_0^{\,n}} = 0.$$

Also,

$$m\frac{d^2w}{dt^2} + 2\mu e(r_0+w)^{-3} + 2\alpha e^2(r_0+w)^{-5} - \beta(r_0+w)^{-n} = 0,$$

or

$$m\frac{d^2w}{dt^2} + w\left(\frac{n\beta}{r_0^{\,n+1}} - \frac{6\mu e}{r_0^{\,4}} - \frac{10\alpha e^2}{r_0^{\,6}}\right) = 0,$$

if powers of $\frac{w}{r_0}$ greater than one are neglected compared with one. The frequency of the simple harmonic motion is therefore

$$\frac{1}{2\pi m^{\frac{1}{2}}}\cdot\left[\frac{\mu e}{r_0^{\,4}}(2n-6) + \frac{\alpha}{r_0^{\,6}}e^2(2n-10)\right]^{\frac{1}{2}}.$$

Hendricks[8] found that the nearest distances Cu—Cl in the complex chlorides of the type $X_2CuCl_4.2H_2O$, where X is an alkali metal, were 2·44, 2·37, and 2·41 Å.U. The cupric ion has therefore a radius of ·6 Å.U. (the radius of Cl^- is 1·81 Å.U.): we take

$$\mu = 1{\cdot}7 . 10^{-18} \text{ E.S.U.} \quad \text{and} \quad \alpha = 1{\cdot}46 . 10^{-24}.$$

This gives the following values for the frequency:—

n	5	6	7
Frequency ...	$4{\cdot}4 . 10^{12}$	$6{\cdot}7 . 10^{12}$	$8{\cdot}3 . 10^{12}$

The value of n is not likely to be high, as the repulsive field on the water dipoles is due not only to the electron sheath of the cupric ion but also to the lattice of sulphate ions. A value $6 \pm 2 . 10^{12}$ is thus probable for the frequency. This is 10^4 times the value calculated from the reaction kinetics.

This discrepancy may be due to various causes. There is no doubt that a term should be included in the reaction rate to allow for the screening effect of the pseudomorph. A molecule of water oscillating at the interface can pass into the pseudomorph only through the interstices of the latter. If after its activated oscillation it collides with an ion of the pseudomorph it will be reflected back. The reaction rate $\frac{n}{S\tau_0}e^{-u_0/kT}$ should therefore be multiplied by a fraction.

There is also the possibility of a time lag between the various stages of the decomposition, *i. e.*, as the four water molecules are successively removed. There will be partial freedom of the water molecules over the cupric ion. The last molecules of the four may have to move to a favourable position before they come off.

If the energy of activation is shared between p degrees of freedom, or, rather, if there are p square terms in the equation for the energy, the reaction rate is

$$\frac{n}{S\tau_0}\cdot\left(\frac{E}{kT}\right)^{\frac{p}{2}-1}\cdot e^{\frac{-E}{kT}}\cdot\frac{1}{\left|\frac{p}{2}-1\right.}{}^{(2)},$$

where

$$E = u_0 + \left(\frac{p}{2}-1\right)kT,$$

and u_0 is measured from the reaction rate. This would make the value of the frequency calculated from the reaction rate even smaller than before for reasonable values of p.

My thanks are due to Mr. Hume and to Dr. Colvin, of this Department, for reading and discussing the manuscript.

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(1) Frenkel, *Zeit. f. Phys.* xxvi. p. 117 (1924).
(2) Hinshelwood, 'Kinetics of Chemical Change in Gaseous Systems,' second edition.
(3) Clausing, *Ann. der Phys.* v. pp. 489, 522 (1930).
(4) Cockroft, Proc. Roy. Soc. A, cxix. p. 293 (1928).
(5) Hume and Colvin, Proc. Roy. Soc. A, cxxv. p. 635 (1929).
(6) Garner and Tanner, J. C. S. p. 47 (1930).
(7) Carpenter and Jette, J. A. C. S. xlv. p. 578 (1923).
(8) Hendricks, J. A. C. S. xlix. p. 2149 (1927).

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XXIV. *The Vibrations of Revolving Shafts.* *By* R. C. J. Howland, *M.A.*, *D.Sc.*, *University College, London* *.

Introduction.

THE phenomenon of instability in a shaft rotating at certain critical speeds is now a very familiar one, and is of recognized importance. It has been the subject of very numerous researches, both theoretical and practical †. The problem of vibration at speeds other than critical has received less attention, but is still a familiar one. The natural frequencies of a uniform shaft at any speed of rotation are well known ‡, while certain phenomena of vibration at speeds both below and above the first whirling speed have been described and theories of them presented. Among these, two are of especial interest from the point of view of the present paper, and will be considered in some detail.

A few years ago there was some controversy concerning the alleged occurrence of large vibrations in shafts run at a speed of $\frac{1}{\sqrt{2}}\Omega_1$, where Ω_1 is the first whirling speed. A theory to account for this was advanced by Kerr § and gave rise to considerable correspondence ||, in the course of whlch vibrations at a speed of $\frac{1}{2}\Omega_1$ were said to be of common occurrence. A theory leading to a vibration speed of $\frac{1}{2}\Omega_1$ was given by Stodola ¶. Subsequent experimental work has favoured his result. Systematic attempts to observe vibrations at a speed of $\frac{1}{\sqrt{2}}\Omega_1$ have failed **, while recent observations suggest that vibrations at the lower speed are usual ††.

The theories offered by Kerr and by Stodola differ fundamentally in the causes they assign to the vibration; according to Kerr it is a resonance phenomenon; Stodola's theory, on

* Commnnicated by the Author.

† A fairly full account of the theory, with some of the more important references, is given by Morris, 'The Strength of Shafts in Vibration' (London: Crosby Lockwood, 1929).

‡ See, for example, Morley, 'Strength of Materials,' § 165.

§ 'Engineering,' vol. ci. pp. 150, 197, 224, 245.

|| *Ibid.* pp. 251, 287, 296, 320, 386, 410, 420, 460, 482, 536.

¶ *Ibid.* p. 386.

** Frith and Buckingham, "The Whirling of Shafts," Journ. Inst. Elec. Eng. lxii. p. 107 (1924).

†† Naylor, Selected Eng. Papers, Inst. C. E. no. 36 (1926); also Brit. Assoc. Report 1927 (abstract only).

the other hand, depends on a lack of balance in the masses attached to the shaft. On this account Stodola's analysis, though it is concerned with a more practically important phenomenon, will not be discussed in the present paper.

Kerr applied his theory to a rather special type of rotor. It is easily shown that it applies equally well to a uniform shaft *, and it will appear in the sequel that a much more general case may be treated, with a similar result. But his theory, though correct in nearly every respect, contains a fallacy. We shall see that, although a *tendency* to vibration at a speed $\frac{1}{\sqrt{2}}\Omega_1$ exists in general, the vibration cannot be actually produced by the cause that he considered sufficient. The theory leads in fact to the conclusion that the phenomenon will be sporadic, and this seems to accord with experience.

The other phenomenon to which some attention will be devoted is that of "shaft-whipping" at speeds above Ω_1. A number of observations of sudden instability of shafts at high speeds were made by Newkirk and Kimball †, and a theory was put forward ‡. In this it was suggested that the instability had its origin in the internal friction of the shaft. This appears to the present writer to be intrinsically unlikely, but some experimental evidence in support of the suggestion was produced. In what follows the effects of internal friction will be considered on the basis of certain not improbable assumptions. The results do not give support to Kimball's theory, and alternative explanations are suggested. No final conclusion is reached. A much more elaborate analysis appears to be needed before the conditions at speeds above Ω_1 can be dealt with adequately.

The Assumptions of the Elementary Theory.

The fundamental assumptions of the elementary theory of the vibrations and whirling of shafts are:

(1) The displacements are everywhere so small that the square of the slope is negligible.

(2) The elementary (Euler-Bernoulli) theory of flexure may be applied.

* See, for example, Morley, 'Strength of Materials,' § 165.

† Newkirk, Gen. Elec. Rev. xxvii. p. 169; Kimball, *ibid.* xxvii. p. 244.

‡ Kimball, *ibid.* xxviii. pp. 554-558.

(3) The moment of inertia of each part of the shaft about its axis is so small that all gyroscopic effects are negligible.

The limitations imposed by these assumptions are considerable, but are not as great as is sometimes supposed. Objections have at times been raised to the theory on the grounds that it neglects effects which, when treated descriptively, appear to be important, but which analysis shows to be of the second order. A brief *résumé* of some of the consequences of assumptions (1), (2), and (3) will therefore be given.

From (1) it follows that the theory can give no satisfactory account of what happens after instability begins. The displacements may then be sufficiently large to make the theory inoperable, and the return of the shaft to stability cannot be explained without less restrictive assumptions. This limitation of the theory is similar to that imposed on the whole theory of small vibrations, but is more serious for a reason which will be explained in the paragraph headed ' Instability as a Dynamical Problem."

Assumption (2) does not necessarily imply (1). In specially simple cases the theory can be worked out without assuming (1) *. Nor does (1) imply (2) ; a more general theory of rotors in which the strains are small is possible, and, in a wide class, the relationship between the natural frequencies at different speeds will be the same as that established below by the use of the simple theory of flexure.

Assumption (3) has two important consequences. In the first place it is easily seen to lead to equations which are the same relative to axes rotating with the shaft as they would be with respect to fixed axes, except that a field of centrifugal force must be introduced. It also leads to the result that the motions in any two mutually perpendicular planes through the axis and revolving with it are sensibly independent. As the equations of motion in these two planes are similar, it is sufficient to consider one plane only. This is usually done without explanation. As a result the theory has at times been criticized on the ground that it assumes the motion to lie entirely in a plane revolving with the shaft. What it really does is to consider only one component of the motion. It is usually unnecessary to do more, but by combining motions in two mutually perpendicular planes we may

* R. von Mises, *Monatshefte f. Math. u. Phys.* p. 44 (1911); E. Schwerin, "Die Stabilität rotierenden achsial belasteter Wellen," *Zeit. f. ang. Math. u. Mech.* v. p. 101 (1925).

obtain any of the forms of more complicated displacement which may occur in practice. For example, a spiral displacement rotating with the shaft is possible, or a displacement of constant magnitude and form rotating at a rate different from the speed of the shaft. The theory does not ignore such possible displacements. It only asserts that they may be resolved into plane components which are effectively independent.

The Vibrations of a Revolving Shaft.

We consider the very general case of a non-uniform shaft under tension or end-thrust, rotating and vibrating. The fixing conditions need not be specified ; it is only necessary to suppose that they are sufficient to make the natural periods determinate when the shaft is not rotating. Motion in one plane through the axis and revolving with it will be considered, and the following notation used :—

Young's modulus	E
Density of material	ρ
Area of cross-section	A
Second moment of cross-section	$I=Ak^2$
End-thrust	P
Angular velocity	ω
Distance from a chosen point of axis. . . .	x
Transverse deflexion from equilibrium position.	y
Frequency of vibrations	σ

Among these E, ρ, A, k, P may all be functions of x without in the least invalidating the succeeding analysis.

The equation of motion of the shaft is easily found to be *

$$\frac{d}{dx}\left(\rho I \frac{d^3y}{dx\,dt^2}\right) = \frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} + \rho A\left(\frac{d^2y}{dt^2} - y\omega^2\right), \quad (1)$$

the term on the left being always negligible †. If an external force $F(x)$ per unit length acts transversely on the shaft the equation will be

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} + \rho A\left(\frac{d^2y}{dt^2} - y\omega^2\right) - F(x). \quad (2)$$

* *Cf.* the author's paper, "The Vibrations of Rods and Shafts with Tension or End-Thrust," Phil. Mag. i. pp. 674–694 (1926).

† Rayleigh, 'Theory of Sound,' i. para. 186.

If the shaft has attached to it a number of loads, such as pulleys, these will be equivalent to local variations of the density and stiffness of the shaft, and equations (1) and (2) will still apply provided that the moments of inertia of the attached masses are not sufficient to invalidate assumption (3) of the previous section. They may make the neglect of the term on the left-hand side of (1) less justified, but its value is likely to remain small *.

To obtain the solution for free vibrations let $F(x)=0$, and write

$$y=\eta\cos(2\pi\sigma t+\alpha).$$

The equation then takes the form

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\eta}{dx^2}\right)+P\frac{d^2\eta}{dx^2}-\rho A(\pi^2\sigma^2+\omega^2)\eta=0. \quad . \quad (3)$$

When the shaft is not rotating this equation has, by supposition, a number of non-zero solutions satisfying the boundary conditions and corresponding to special values of σ, the natural frequencies. Denote these by $\sigma_1, \sigma_2, \ldots, \sigma_r, \ldots$. It is obvious that the general equation (3) will have the same non-zero solutions corresponding to values σ' of σ given by

$$4\pi^2\sigma_r'^2+\omega^2=4\pi^2\sigma_r^2,$$

i.e.,

$$\sigma_r'=\left\{\sigma_r^2-\frac{\omega^2}{4\pi^2}\right\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

As ω increases each σ' decreases. When ω rises to $2\pi\sigma_1$ σ_1' becomes zero, and any displacement of the type corresponding to σ_1 tends to be permanent. This value of ω is the first whirling speed. Denoting it by Ω_1, we may write the equation for the lowest frequency in the form

$$2\pi\sigma_1'=\{\Omega_1^2-\omega^2\}^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Resonance Phenomena.

We next consider the forced vibrations corresponding to any periodic disturbing force of frequency n. This may be represented by taking the function $F(x, t)$ in equation (2) to have the form $f(x)\cos 2\pi nt$. Let a solution of (3) corresponding to σ_r' be

$$A_rX_r\cos(2\pi\sigma_r't+\alpha_r),$$

* For a discussion of the effect on the first whirling speed see Hahn, "Note sur la Vitesse Critique des Arbres et la Formule de Dunkerley," *Schweiz. Bauz.* Nov. 1918.

where A_r and α_r are arbitrary constants and X_r is a function of x. Then the complementary function of (2) is

$$\sum_{r=1}^{\infty} A_r X_r \cos(2\pi\sigma_r' t + \alpha_r).$$

It will in general be possible to expand $f(x)$ in a series of the functions X_r *. The equation will then take the form

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} + \rho A\left(\frac{d^2y}{dt^2} - y\omega^2\right)$$

$$= \cos 2\pi nt \sum_{r=1}^{\infty} B_r X_r, \quad . \quad (6)$$

where the B_r are constants depending on the form of $f(x)$.

To obtain a particular integral of this equation write

$$y = \sum_{r=1}^{\infty} C_r X_r \cos 2\pi nt. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Then, remembering that by the definition of X_r

$$\frac{d^2}{dx^2}\left(EI\frac{d^2X_r}{dx^2}\right) + P\frac{d^2X_r}{dx^2} = \rho A . 4\pi^2\sigma_r^2 X_r, \quad . \quad . \quad (8)$$

we get

$$\sum_{r=1}^{\infty} \rho A C_r X_r \{4\pi^2\sigma_r^2 - 4\pi^2 n^2 - \omega^2\} \cos 2\pi nt$$

$$= \sum_{r=1}^{\infty} B_r X_r \cos 2\pi nt,$$

which is satisfied if

$$C_r = \frac{B_r}{\rho A\{4\pi^2\sigma_r^2 - \omega^2 - 4\pi^2 n^2\}}$$

$$= \frac{B_r}{4\pi^2\rho A(\sigma_r'^2 - n^2)}.$$

Adding the complementary function, we now have for the complete solution

$$y = \sum_{n=1}^{\infty}\left[\frac{B_r X_r}{4\pi^2\rho A(\sigma_r'^2 - n^2)}\cos 2\pi nt + A_r \cos(2\pi\sigma_r' t + \alpha_r)\right], \quad . \quad . \quad . \quad (9)$$

where the constants A_r and α_r have to be determined from

* Rayleigh, 'Theory of Sound,' discusses such expansions from the point of view of the applied mathematician. For a brief summary see the author's paper "Transverse Oscillations in Girders," Selected Papers, Inst. C. E. no. 23 (1924).

the initial conditions. For example, if there is no disturbance up to the time $t=0$, the solution will be

$$y=\sum_{n=1}^{\infty}\frac{B_rX_r}{4\pi^2\rho A(\sigma_r'^2-n^2)}\{\cos 2\pi nt-\cos 2\pi\sigma_r't\}. \quad (10)$$

If n is in the neighbourhood of σ_r' the disturbance will be large compared with the force producing it. This is the phenomenon of *Resonance.* It may occur at any speed of rotation if a disturbing force is present and if its frequency coincides with one of the natural frequencies of the shaft at that speed. It is most likely to be effective if it coincides in frequency with σ_1', as the vibration of lowest frequency is always the easiest to excite. To determine in advance the possible speeds at which resonance may appear it is necessary to have a knowledge of all the disturbing forces that can act on the shaft, and this will not in general be practicable. But the most effective disturbances are likely to be those produced by the machine of which the shaft forms a part. Periodic motions in the machine will be likely to have periods which are simple multiples or submultiples of the period of rotation of the shaft, so that resonance is likely to occur at speeds for which ω^2 bears a simple ratio to Ω_1^2.

If the disturbing force has the same period as the shaft, *i. e.*, if $n=\frac{\omega}{2\pi}$, resonance will occur at a speed given by

$$\sigma_1'^2=\frac{\omega^2}{4\pi^2},$$

or

$$\sigma_1^2=\frac{\omega^2}{4\pi^2}+\frac{\omega^2}{4\pi^2}=\frac{\omega^2}{2\pi^2}.$$

Putting

$$2\pi\sigma_1=\Omega_1,$$

we get

$$\omega=\frac{1}{\sqrt{2}}\Omega_1, \quad \dots\dots\dots \quad (11)$$

and this is the vibration speed predicted by Kerr. It is to be observed, however, that resonance will occur at this speed only if there exists a disturbing force of the right period. Kerr supposed that the weight of the shaft itself provided such a force. It is evident that the component of the weight in any plane rotating with the shaft appears as a periodic force of frequency equal to that of the shaft's rotation; but this force is not effective, since it is balanced from the start by the initial displacement of the shaft. The course of

events is as follows :—first the shaft bends slightly under its own weight and takes up a new equilibrium position ; then it rotates about its bent axis ; finally, if there is a vibration, this takes place about the equilibrium position and not about the unstrained axis. The weight remains balanced throughout. This is sufficiently obvious; but it was lost sight of not only by Kerr and his supporters but also by their critics *.

Further Generalization of the Theory.

Equation (4) for the frequencies of a revolving shaft and its special form (5) are easily seen to be more general than has so far been proved. They depend only upon the possibility of relating the vibrations to a linear *integral* equation, and this may be much more general than the linear differential equation (2). The only assumptions needed are:

(i.) the shaft has determinate modes of vibration when not rotating ;

(ii.) a transverse force acting at any point of the shaft produces at any other point a transverse deflexion proportional to the force.

Let the deflexion produced at a point whose abscissa is α by a unit force at a point whose abscissa is x be denoted by $K(x, \alpha)$. It is a consequence of Rayleigh's general reciprocal theorem that $K(x, \alpha)$ is symmetrical in its two variables †, but this is not relevant to the present discussion except in so far as it ensures that the integral equation shall be of the right type ‡.

If the deflexion of the axis at any instant is y_x we may, applying D'Alembert's Principle, say that this deflexion could be maintained as a static deflexion by the external forces and the centrifugal force together with the reversed effective forces (mass-accelerations). On an element of length δx the forces acting will then be :

(i.) an external force $F(x, t)\delta x$;

(ii.) a centrifugal force $\rho A y_x \omega^2 \delta x$;

* Rodgers, "On the Vibration and Critical Speeds of Rotors," Phil. Mag. xliv. pp. 122–156 (1922), states that a vibration at a speed $\frac{1}{\sqrt{2}}\Omega_1$ can only be produced by *external* forces.

† 'Theory of Sound,' i. para. 72.

‡ For a treatment of integral equations from the standpoint of applied mathematics see Frank u. von Mises, 'Differentialgleichungen der Physik,' Braunschweig (1930).

(iii.) the reversed effective force $-\rho A\delta x \dfrac{d^2y_x}{dt^2}$.

We shall consider only free vibrations in which (i.) is absent. The deflexion produced at the point of abscissa α by the forces on the element δx is then

$$\rho A\delta x \left\{\omega^2 y_x - \frac{d^2y_x}{dt^2}\right\} K(x, \alpha),$$

and the whole deflexion will be the integral of this over the length of the shaft. This must, however, be the actual deflexion at the point. Hence

$$y_\alpha = \int_0^l \rho A \left\{\omega^2 y_x - \frac{d^2y_x}{dt^2}\right\} K(x, a)\, dx.$$

For vibrations of frequency σ put

$$y_x = f(x) \cos(2\pi\sigma t + \epsilon).$$

Then the equation becomes *

$$f(\alpha) = (4\pi^2\sigma^2 + \omega^2) \int_0^l \rho A f(x) K(x, \alpha)\, dx. \quad . \quad (12)$$

By supposition this equation has certain solutions when $\omega = 0$ corresponding to special values σ_r of σ. Obviously it will have the same solutions in general, but corresponding to values σ_r' of σ given by the equation

$$4\pi^2\sigma_r'^2 + \omega^2 = 4\pi^2\sigma_r^2.$$

This brings us back to equation (4), from which (5) follows immediately in the same way as before.

Effects of Internal Friction.

The laws of internal friction, or viscosity, in solids are very imperfectly known. In any attempt at a mathematical treatment of its effects a law must be chosen which is analytically convenient while representing the facts in their broader outline. For our present purpose laws expressed in terms of integrals over a time-cycle † are not convenient, as it is difficult to incorporate them into our differential equations. One law of a suitable type was suggested by

* *Cf.* the author's paper "Application of an Integral Equation to the Whirling Speeds of Shafts," Phil. Mag. iii. pp. 513–528 (1927).

† Among many papers in which a law of this type is used may be mentioned G. H. Keulegan, "Statical Hysteresis in the Flexure of Bars," Bureau of Standards, Wash. xxi. pp. 145–162 (1926).

Filon and Jessop * in a form suitable for application to elastic solids in general. It was applied by Miss Hosali †, and later by Sezawa ‡, to a discussion of earthquake waves, and by Suyehiro § to the vibrations of bars ‖.

In the simplified form required for our present purpose it may be taken as stating that in an elastic fibre undergoing extension the tension consists of two parts—a principal part proportional to the instantaneous extension and a smaller part proportional to the rate of extension.

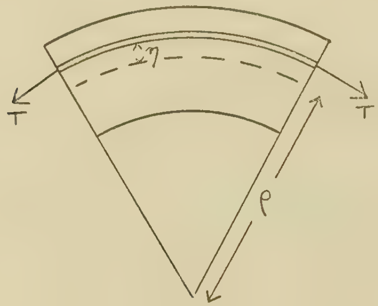
Consider an element of the shaft bent according to the usual Euler-Bernoulli assumptions (fig. 1).

Writing

$\rho=$ radius of curvature of neutral fibre after bending,

$\eta=$ distance of typical fibre from neutral axis,

Fig. 1.



the strain in the fibre is η/ρ. Hence the stress in the fibre is

$$\mathrm{E}\frac{\eta}{\rho}+k\eta\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right),$$

where k is a small constant depending on the viscosity ¶.

* "On the Stress-Optical Effect in Transparent Solids strained beyond the Elastic Limit," Proc. Roy. Soc. A, ci. pp. 165-169 (1922).

† "On Seismic Waves in a Visco-Elastic Earth," Proc. Roy. Soc. A, civ. pp. 271-278 (1923).

‡ "On the Decay of Waves in Visco-Elastic Solid Bodies," Bull. Earthq. Res. Inst. Tokyo, iii. pp. 43-54 (1927).

§ "On the Upper Limit of the Frequency of the Transversal Vibration of Prismatic Bars," Proc. Imp. Acad. Tokyo, iv. pp. 263-266 (1928).

‖ There are alternative hypotheses which, in the simple case of Euler-Bernoulli flexure, are not distinguishable from that of Filon and Jessop, which, for the present purpose, may be regarded merely as typical of a class.

¶ The other symbols introduced here and in the next few lines are conventional in the elementary theory of bending, and will not be explained.

Multiplying this by η, and integrating over the cross-section, we get for the bending moment

$$M=\frac{EI}{\rho}+kI\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right)=EI\frac{\partial^2 y}{\partial x^2}+kI\frac{\partial^3 y}{\partial t\partial x^2}. \quad . \quad (13)$$

The theory now continues on familiar lines, leading to the vibration equation

$$\rho A\left(\omega^2 y-\frac{\partial^2 y}{\partial t^2}\right)=\frac{\partial^2 M}{\partial x^2},$$

or

$$\frac{\partial^2}{\partial x^2}\left[EI\frac{\partial^2 y}{\partial x^2}+kI\frac{\partial^3 y}{\partial t\,\partial x^2}\right]+\rho A\left(\frac{\partial^2 y}{\partial t^2}-\omega^2 y\right)=0. \quad (14)$$

In this the end thrust has been ignored, as its introduction complicates the analysis considerably without adding any features of interest.

To obtain solutions of this equation write

$$y=\eta e^{2\pi i\sigma t},$$

where η is a function of x only. Then, supposing that E and k are independent of x, the equation for η may be written

$$\frac{d^2}{dx^2}\left[I\frac{d^2\eta}{dx^2}\right]=m^4\eta, \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where

$$m^4=\frac{\rho A(4\pi^2\sigma^2+\omega^2)}{E+2\pi i\sigma k}, \quad . \quad . \quad . \quad . \quad (16)$$

Now equation (15) has the same form for all values of k and ω. It is easily seen that the form of the boundary conditions for η will also be the same. Hence any solution valid for one pair of values of k and ω will remain valid for a different pair of values provided that we can choose a new value of σ in such a way that m is unchanged. Suppose then that when k and ω are both zero the shaft has a number of determinate modes of vibration. Let one of these be given by

$$y=X_r\cos(2\pi\sigma_r t+\alpha_r).$$

Then equation (15) is satisfied by $\eta=X_r$, with

$$m^4=4\pi^2\rho A\sigma_r^2/E.$$

It follows that in the general case, when k and ω are not zero, it will still be satisfied by $\eta=X_r$, provided that σ has a value given by

$$\frac{\rho A(4\pi^2\sigma^2+\omega^2)}{E+2\pi i\sigma k}=\frac{\rho A(4\pi^2\sigma_r^2)}{E},$$

or

$$4\pi^2\sigma^2+\omega^2=4\pi^2\sigma_r^2(1+2\pi ik'\sigma), \quad . \quad (17)$$

where $$k' = \frac{k}{E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (18)$$

Since k' is small this equation is best solved by successive approximation. For a first approximation take $k'=0$. Then

$$4\pi^2\sigma^2 + \omega^2 = 4\pi^2\sigma_r^2,$$

or $$\sigma = \sigma_r',$$

with the same meaning for σ_r' as in (4).

For the second approximation we insert the first approximation to σ in the right-hand side of (17). Then

$$\sigma^2 = \sigma_r'^2 + 2\pi i k' \sigma_r^2 \sigma_r',$$

from which $$\sigma = \sigma_r' + 2\pi i k' \sigma_r^2, \quad . \quad . \quad . \quad . \quad . \qquad (19)$$

neglecting the square of k'.

We now have a solution of (14) in the form

$$y = X_r e^{2\pi i \sigma_r' t - 4\pi^2 k' \sigma_r^2 t}.$$

The real and imaginary components will separately be solutions, and by combining them we can obtain a solution of the form

$$y = X_r e^{-4\pi^2 k' \sigma_r^2 t} \cos(2\pi\sigma_r' t + \alpha). \quad . \quad . \quad . \quad . \qquad (20)$$

More generally a solution is

$$y = \Sigma A_r X_r e^{-4\pi^2 k' \sigma_r^2 t} \cos(2\pi\sigma_r' t + \alpha_r), \quad . \qquad (21)$$

the summation extending to all the values of r to which vibrations of the shaft with k' and ω zero correspond. It is, moreover, obvious that (21) is the general solution, for, by hypothesis, we started with the complete set of modes of vibration of the shaft with k' and ω zero, and hence an arbitrary initial displacement of the shaft can be expanded in terms of the functions X_r. Thus the constants A_r in (21) may be determined to correspond with arbitrary initial conditions.

The result of this section is to show that, in the very general case of a non-uniform shaft with arbitrary conditions, the only effect of internal friction is to introduce damping factors into the component vibrations. The modes and their frequencies are unaffected. No new types of instability can therefore be introduced.

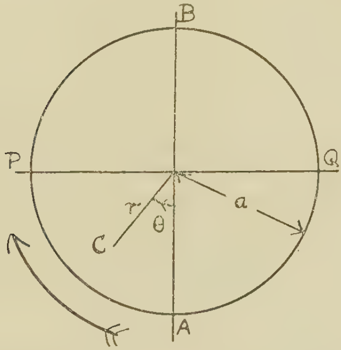
There is, however, another effect of internal friction which is conveniently illustrated by considering the result of combining its action with that of the weight of the shaft. Let the circle in fig. 2 represent a section of the shaft at any

point of its length, A being the lowest point. In consequence of the permanent deflexion produced by the weight the fibres are in tension at A and in compression at the highest point B. If e be the strain at A, the strain at any other point C will be $(er/a)\cos\theta$, and will be changing at a rate $-(er/a)\omega\sin\theta$. This will, on account of viscosity, produce in a polar element of area $r\,dr\,d\theta$ at C a compressive force of magnitude $(er/a)\omega k\sin\theta\,.\,r\,dr\,d\theta$. Taking moments about the axes PQ, AB respectively, and integrating over the section, we obtain bending moments

$$\mathrm{M}_1=\frac{e\omega k}{a}\int_0^a r^3\,dr\int_0^{2\pi}\sin\theta\cos\theta\,d\theta=0\,;$$

$$\mathrm{M}_2=\frac{e\omega k}{a}\int_0^a r^3\,dr\int_0^{2\pi}\sin^2\theta\,d\theta=\pi\omega k a^3 e.$$

Fig. 2.



Now e is proportional to the bending moment produced about PQ by the weight. Hence there is a small bending moment about AB of a kind which would be produced if a small fraction of the weight of the shaft were acting in the direction PQ. The shaft has a deflexion in a plane slightly inclined to the vertical, as if the direction of gravity were deflected.

The theory of "shaft-whipping" put forward by Kimball appears to rest on the idea that a deflecting force of the kind just established can produce a whirling motion of the shaft by acting always at right angles to the instantaneous deflexion of the shaft. But this idea seems to contain two distinct fallacies. In the first place, the force can act continuously in one sense only if there is a steady force, such as the weight, to keep the shaft deflected, and in this case, as has been seen, the result is only a slight permanent change

of deflexion. Any deflexion of the shaft not maintained by a steady force would give rise to vibrations, and the bending moments produced by viscosity would also be oscillatory. At most then the effect would only be to modify an already existing motion. Instability can occur only if one of the natural periods of the shaft tends to infinity, or if there is resonance between a natural period and an external force. The periods themselves are not altered by viscosity, while any oscillatory forces produced by it will be isochronous with an already existing vibration, and could only produce resonance if it is already there.

But even this is to claim too much for its powers. On the elementary theory at least any vibration will be the sum of components each of which rotates with the shaft, and it is clear that the effect under consideration will vanish in this case. The whole truth will be contained in equation (14), whose consequences have already been seen. The reason for instability at speeds above Ω_1 must be sought in some other direction.

Stability as a Dynamical Problem.

The treatment of stability problems by statical methods, though often convenient, is artificial, and often leads to paradoxical results. For example, the usual theory of the stability of long struts leads to the conclusion that the strut will be stable under any thrust unless this has one of a series of critical values. According to the theory, if a thrust greater than the least critical thrust but less than the second were applied the strut would remain stable.

Stability is not, however, a question of the possibility of statical configurations. The question to be answered is whether an arbitrary deflexion of the system will lead to small vibrations about the equilibrium position or whether the system will move away from its initial position entirely. Treated from this point of view the theory of struts leads to the conclusion that there is instability for all thrusts greater than the least critical thrust *.

An exactly similar result applies to a rotating shaft. Equation (5) shows that when ω exceeds Ω_1 the value of σ_1' is imaginary and the time factor associated with a deflexion of type X_1 will be a real exponential. According to the elementary theory, therefore, the shaft is unstable at all high speeds. What is in need of explanation is not its

* A simple case was treated by Cowley and Levy in the course of another investigation, "Vibrations and Strength of Struts and Continuous Beams under End-Thrusts," Proc. Roy. Soc. (1918). No further solutions appear to have been published.

occasional instability but its much more frequent stability. This would involve the production of a theory free from at least some of the restrictions of the elementary theory. Some specially simple cases have been dealt with by other writers *. No attempt to extend their methods will be made here. It may be said, however, in criticism of their results, that they treat the problem as a static one. Their conclusion that the shaft is necessarily stable at speeds above Ω_1 cannot be accepted, even for the simple cases treated, without reserve.

Instability at High Speeds.

The phenomenon of "shaft whipping" at high speeds remains unexplained, but the foregoing discussion suggests two possible directions in which an explanation may be sought. It is, first, a possibility that the phenomenon is one of resonance with some motion in the machine. There are a number of possible speeds at which the value of σ_r' given by (4) will coincide with some simple multiple of ω, and is therefore likely to coincide with the frequency of some other moving part. The descriptions given of the effect are, however, strongly suggestive of true instability rather than resonance. This suggests, especially if, as has been averred, the instability occurs at no special speed, that the result of the last paragraph is really significant and there is instability at all speeds above Ω_1, but with some influence preventing it from making its appearance under normal circumstances. This influence must be sought among those neglected in the theory. It may be the gyroscopic effect or it may be the non-linear part of the elasticity. The suggestion here to be made is that both of these are important, but not at the same time.

The non-linear terms in the elasticity will tend to be more important when the deflexion is large. The gyroscopic influence may well, however, have important effects upon small deflexions when the speed of the shaft is high. It is suggested then that the gyroscopic effects are sufficient to maintain stability for small deflexions, but that, if the deflexion is too great, the instability indicated by the theory comes into effect. A large deflexion is produced, but is finally stabilized again by the action of the non-linear part of the elasticity or by friction. This suggestion is only tentative; it needs support from both analysis and experiment before it can be accepted.

* R. von Mises and E. Schwerin, *loc. cit. ante.* Both writers deal with a massless shaft with one central mass attached.

XXV. *Stresses due to a small Elliptic Hole or a Crack on the Neutral Axis of a Deep Beam under Constant Bending Moment.* *By* Bibhutibhusan Sen, *M.Sc.* *

§ 1. *Introduction.*

THE effect of a small circular hole on the neutral axis of a very deep beam of rectangular section has been found by Z. Tuzi †, who has verified the theoretical results experimentally by photoelastic methods. An elliptic hole or a crack in an infinite plate under uniform tension has been the subject of many researches ‡. The object of the present paper is to examine theoretically the stresses due to a small elliptic hole with its centre on the neutral axis of a deep beam under uniform bending moment. First, the stresses due to an elliptic hole with its major axis inclined at an angle θ to the neutral axis have been determined, and then the simpler cases of the elliptic hole with its major axis along and perpendicular to the neutral axis have been deduced. Though the method applied can be extended to any uniform beam, only the case of a deep beam of rectangular section has been considered here.

If the centre of the hole be the origin, the neutral axis the axis of x, and a line at right angles to it through the origin the axis of y, then, since the problem of a deep beam can be considered as that of plane stress, the stresses can be expressed in terms of the single function χ such that

$$\widehat{xx}=\frac{\partial^2\chi}{\partial y^2}; \quad \widehat{yy}=\frac{\partial^2\chi}{\partial x^2}; \quad \widehat{xy}=-\frac{\partial^2\chi}{\partial x\,\partial y}, \quad . \quad . \quad (1)$$

χ satisfying the relation

$$\left(\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}\right)\chi=0. \quad . \quad . \quad . \quad (2)$$

§ 2. *The Elliptic Hole with its Major Axis inclined at an Angle θ to the Neutral Axis.*

Let the elliptic boundary be denoted by $\xi=\alpha$ of the series of curves given by the relation

$$x'+iy'=c\cosh(\xi+i\eta), \quad . \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

† "Effect of a Circular Hole on the Stress Distribution in a Deep Beam under Uniform Bending Moment," by Z. Tuzi, Phil. Mag. ser. 7, vol. ix. p. 210 (1930).

‡ *Cf.* "The Phenomenon of Rupture and Flow in a Solid," by A. Griffith, Trans. Roy. Soc. Lond. vol. ccxxi. In this paper other references will be found.

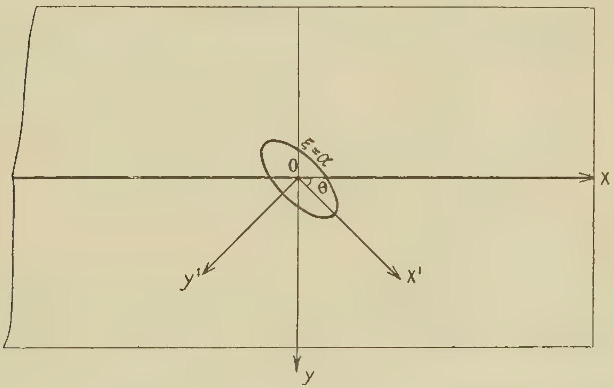
ox' and oy' being the directions of the major and the minor axis as shown in the figure.

The stress components $\widehat{\xi\xi}$, $\widehat{\xi\eta}$, and $\widehat{\eta\eta}$ are then given by the relations

$$\widehat{\xi\xi} = \frac{c^2h^4}{2}\left[(\cosh 2\xi - \cos 2\eta)\frac{\partial^2\chi}{\partial\eta^2} + \sinh 2\xi\frac{\partial\chi}{\partial\xi} - \sin 2\eta\frac{\partial\chi}{\partial\eta}\right],$$

$$\widehat{\eta\eta} = \frac{c^2h^4}{2}\left[(\cosh 2\xi - \cos 2\eta)\frac{\partial^2\chi}{\partial\xi^2} - \sinh 2\xi\frac{\partial\chi}{\partial\xi} + \sin 2\eta\frac{\partial\chi}{\partial\eta}\right],$$

$$\widehat{\xi\eta} = \frac{c^2h^4}{2}\left[-(\cosh 2\xi - \cos 2\eta)\frac{\partial^2\chi}{\partial\xi\,\partial\eta} + \sinh 2\xi\frac{\partial\chi}{\partial\eta} + \sin 2\eta\frac{\partial\chi}{\partial\xi}\right], \quad . \quad . \quad . \quad (4)$$



where

$$\frac{1}{h^2} = \left(\frac{\partial x'}{\partial\xi}\right)^2 + \left(\frac{\partial y'}{\partial\xi}\right)^2 = \frac{c^2}{2}(\cosh 2\xi - \cos 2\eta).$$

Let M be the bending moment, t the thickness, and d the depth of the beam. Then the boundary conditions to be satisfied are

$$\widehat{xx} = \mathrm{A}y, \quad \widehat{yy} = 0, \quad \widehat{xy} = 0 \quad \text{at an infinite distance,} \qquad (5)$$

and

$$\widehat{\xi\xi} = 0, \quad \widehat{\xi\eta} = 0 \quad \text{when} \quad \xi = \alpha, \quad . \quad . \quad . \quad (6)$$

where

$$\mathrm{A} = \frac{12\mathrm{M}}{td^3}.$$

The equation (2) and the condition (5) are satisfied if we assume for the stress function

$$\chi_0 = \frac{A}{6} y^3 = \frac{A}{6}(x' \sin\theta + y' \cos\theta)^3$$

$$= p_0[(3\cosh^3\xi \sin^3\theta + 3\sinh^2\xi \cosh\xi \cos^2\theta \sin\theta)\cos\eta$$
$$+ (\cosh^3\xi \sin^3\theta - 3\sinh^2\xi \cosh\xi \cos^2\theta \sin\theta)\cos 3\eta$$
$$+ (3\sinh^3\xi \cos^3\theta + 3\cosh^2\xi \sinh\xi \sin^2\theta \cos\theta)\sin\eta$$
$$+ (3\cosh^2\xi \sinh\xi \sin^2\theta \cos\theta - \sinh^3\xi \cos^3\theta)\sin 3\eta] \quad (7)$$

$$\left(p_0 \text{ standing for } \frac{A}{24} c^3\right).$$

Let $$\chi = \chi_0 + \chi_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where χ_1 is a solution of the equation (2) and is such that the stresses calculated from it are zero at an infinite distance. Such an expression for the stress function can be written as

$$\chi_1 = p_0[(C_1 e^{\xi} + C_2 e^{-\xi} + C_3 e^{-3\xi})\cos\eta + (C_3 e^{-\xi} + C_4 e^{-3\xi})\cos 3\eta$$
$$+ (D_1 e^{\xi} + D_2 e^{-\xi} + D_3 e^{-3\xi})\sin\eta + (D_3 e^{-\xi} + D_4 e^{-3\xi})\sin 3\eta, \quad . \quad . \quad . \quad (9)$$

where C_1, C_2, C_3, C_4, D_1, D_2, D_3, and D_4 are constants.

Since h^4 is of the order $\frac{1}{e^{4\xi}}$ when ξ is very great, it is apparent that the stresses calculated from (9) will vanish at at infinite distance.

Then $$\chi = \lambda_1 \cos\eta + \lambda_3 \cos 3\eta + \mu_1 \sin\eta + \mu_3 \sin 3\eta, \quad . \quad (10)$$
where

$$\lambda_1 = p_0[C_1 e^{\xi} + C_2 e^{-\xi} + C_3 e^{-3\xi}$$
$$+ 3\cosh^3\xi \sin^3\theta + 3\sinh^2\xi \cosh\xi \cos^2\theta \sin\theta],$$

$$\lambda_3 = p_0[C_3 e^{-\xi} + C_4 e^{-3\xi}$$
$$+ \cosh^3\xi \sin^3\theta - 3\sinh^2\xi \cosh\xi \cos^2\theta \sin\theta],$$

$$\mu_1 = p_0[D_1 e^{\xi} + D_2 e^{-\xi} + D_3 e^{-3\xi}$$
$$+ 3\sinh^3\xi \cos^3\theta + 3\cosh^2\xi \sinh\xi \sin^2\theta \cos\theta],$$

$$\mu_3 = p_0[D_3 e^{-\xi} + D_4 e^{-3\xi}$$
$$+ 3\cosh^2\xi \sinh\xi \sin^2\theta \cos\theta - \sinh^3\xi \cos^3\theta]. \quad . \quad . \quad . \quad (11)$$

The values of the stresses are

$$\frac{2\widehat{\xi\xi}}{c^2h^4} = [-2\lambda_1 \sinh^2\xi + \lambda_1' \sinh 2\xi + 6\lambda_3] \cos\eta$$
$$+ [-9\lambda_3 \cosh 2\xi + \lambda_3' \sinh 2\xi] \cos 3\eta + 3\lambda_3 \cos 5\eta$$
$$+ [-2\mu_1 \cosh^2\xi + \mu_1' \sinh 2\xi + 6\mu_3] \sin\eta$$
$$+ [-9\mu_3 \cosh 2\xi + \mu_3' \sinh 2\xi] \sin 3\eta + 3\mu_3 \sin 5\eta\ ;$$

$$\frac{2\widehat{\eta\eta}}{c^2h^4} = [\lambda_1'' \cosh 2\xi - \lambda_1' \sinh 2\xi - \tfrac{1}{2}(\lambda_1 + \lambda_1'' + 3\lambda_3 + \lambda_3'')] \cos\eta$$
$$+ [\lambda_3'' \cosh 2\xi - \lambda_3' \sinh 2\xi + \tfrac{1}{2}(\lambda_1 - \lambda_1'')] \cos 3\eta$$
$$+ \tfrac{1}{2}(3\lambda_3 - \lambda_3'') \cos 5\eta$$
$$+ [\mu_1'' \cosh 2\xi - \mu_1' \sinh 2\xi + \tfrac{1}{2}(\mu_1 + \mu_1''$$
$$- 3\mu_3 - \mu_3'')] \sin\eta$$
$$+ [\mu_3'' \cosh 2\xi - \mu_3' \sinh 2\xi + \tfrac{1}{2}(\mu_1 - \mu_1'')] \sin 3\eta$$
$$+ \tfrac{1}{2}(3\mu_3 - \mu_3'') \sin 5\eta\ ;$$

$$\frac{2\widehat{\xi\eta}}{c^2h^4} = [2\lambda_1' \cosh^2\xi - 2\lambda_3' - \lambda_1 \sinh 2\xi] \sin\eta$$
$$+ 3[\lambda_3' \cosh 2\xi - \lambda_3 \sinh 2\xi] \sin 3\eta - \lambda_3' \sin 5\eta$$
$$+ [-2\mu_1' \sin^2\xi + 2\mu_3' + \mu_1 \sinh 2\xi] \cos\eta$$
$$+ 3[\mu_3 \sinh 2\xi - \mu_3' \cosh 2\xi] \cos 3\eta + \mu_3' \cos 5\eta. \quad \ldots \quad (12)$$

Here accented letters denote the differential coefficients of the corresponding functions with respect to ξ, double accents denoting the second differential coefficient.

It is evident that the stresses $\widehat{\xi\xi}$ and $\widehat{\xi\eta}$ will vanish on the elliptic boundary if, when $\xi = \alpha$,

$$\lambda_1 = \lambda_1' = \lambda_3 = \lambda_3' = 0$$

and

$$\mu_1 = \mu_1' = \mu_3 = \mu_3' = 0.$$

These conditions give

$$C_1 = -3 \sin\theta [\cosh^2\alpha \sin^2\theta + \sinh^2\alpha \cos^2\theta],$$

$$C_2 = \frac{3}{2} e^{2\alpha} \sin\theta [\cosh^2\alpha \sin^2\theta$$
$$+ \cos^2\theta \sinh\alpha\, (3 \sinh\alpha - 2\cosh\alpha)],$$

$$C_3 = -\frac{3}{2} e^{2\alpha} \sin\theta [\cosh^2\alpha \sin^2\theta$$
$$- \cos^2\theta \sinh\alpha\, (2\cosh\alpha + \sinh\alpha)],$$

$$C_4=\frac{e^{3\alpha}}{2}\sin\theta[\cosh^2\alpha\sin^2\theta(\cosh\alpha+3\sinh\alpha)$$
$$-3\cos^2\theta\sinh\alpha\,(\sinh\alpha\cosh\alpha+\sinh^2\alpha+2\cosh^2\alpha)];$$

$$D_1=-3\cos\theta[\sinh^2\alpha\cos^2\theta+\cosh^2\alpha\sin^2\theta],$$

$$D_2=\frac{3}{2}e^{2\alpha}\cos\theta[\sinh^2\alpha\cos^2\theta$$
$$+\sin^2\theta\cosh\alpha\,(3\cosh\alpha-2\sinh\alpha)],$$

$$D_3=\frac{3}{2}e^{2\alpha}\cos\theta[\sinh^2\alpha\cos^2\theta$$
$$-\cosh\alpha\sin^2\theta(2\sinh\alpha+\cosh\alpha)],$$

$$D_4=\frac{e^{3\alpha}}{2}\cos\theta[3\sin^2\theta\cosh\alpha\,(\sinh\alpha\cosh\alpha+\cosh^2\alpha$$
$$+2\sinh^2\alpha)-\sinh^2\alpha\cos^2\theta(\sinh\alpha+3\cosh\alpha)]. \quad . \quad . \quad . \quad (13)$$

The values of stresses can now be deduced from (11) and (12).

The most important result is the value of $\widehat{\eta\eta}$ on the elliptic boundary $\xi=\alpha$, which is given by

$$\widehat{\eta\eta}=\frac{2}{c^2(\cosh 2\alpha-\cos 2\eta)}\left(\frac{\partial^2 x}{\partial\xi^2}\right)_{\xi=\alpha}$$
$$=\frac{12p_0(\cosh\alpha+\sinh\alpha)}{c^2(\cosh 2\alpha-\cos 2\eta)}[\{(3\sinh\alpha-\cosh\alpha)\cosh\alpha\sin^2\theta$$
$$+(3\sinh\alpha\cosh\alpha+\cosh^2\alpha-2\sinh^2\alpha)\cos^2\theta\}\sin\theta\cos\eta$$
$$+(\cosh\alpha+\sinh\alpha)\{\cosh\alpha\sin^2\theta$$
$$-(2\sinh\alpha+\cosh\alpha)\cos^2\theta\}\sin\theta\cos 3\eta$$
$$+\{(3\cosh\alpha-\sinh\alpha)\sinh\alpha\cos^2\theta$$
$$+(3\cosh\alpha\sinh\alpha+\sinh^2\alpha-2\cosh^2\alpha)\sin^2\theta\}\cos\theta\sin\eta$$
$$+(\cosh\alpha+\sinh\alpha)\{(2\cosh\alpha+\sinh\alpha)\sin^2\theta$$
$$-\sinh\alpha\cos^2\theta\}\cos\theta\sin 3\eta]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

If $\alpha=0$, *i. e.*, if the hole becomes a crack coinciding with the line of foci, then $\widehat{\eta\eta}$ on the crack surface is equal to

$$\frac{Ac}{2}\sin\theta\,\mathrm{cosec}\,\eta[\cos 2\theta\sin 2\eta+\sin 2\theta\cos 2\eta]. \qquad (15)$$

When $\eta=n\pi$ the value of the above expression becomes infinite, and hence the values of this stress at the extremities of the crack are very great.

At the points on the crack given by $\eta=\frac{n\pi}{2}-\theta$ there is no stress.

§ 3. *The Major Axis of the Elliptic Hole along the Neutral Axis of the Beam.*

In this case $\theta=0$, and hence,

$$\left.\begin{aligned} &C_1=C_2=C_3=C_4=0,\\ &D_1=-3\sinh^2\alpha,\\ &D_2=D_3=\frac{3}{2}e^{2\alpha}\sinh^2\alpha,\\ &D_4=-\frac{e^{3\alpha}}{2}\sinh^2\alpha\,(\sinh\alpha+3\cosh\alpha).\end{aligned}\right\} \quad . \quad . \quad (16)$$

The stresses are given by

$$\begin{aligned} \frac{2\widehat{\xi\xi}}{c^2h^4}&=[-2\mu_1\cosh^2\xi+\mu_1'\sinh 2\xi+6\mu_3]\sin\eta\\ &\quad+[-9\mu_3\cosh 2\xi+\mu_3'\sinh 2\xi]\sinh 3\eta+3\mu_3\sin 5\eta\,;\\ \frac{2\widehat{\eta\eta}}{c^2h^4}&=[\mu_1''\cosh 2\xi-\mu_1'\sinh 2\xi+\tfrac{1}{2}(\mu_1+\mu_1''-3\mu_3'')]\sin\eta\\ &\quad+[\mu_3''\cosh 2\xi-\mu_3'\sinh 2\xi+\tfrac{1}{2}(\mu_1-\mu_1'')]\sin 3\eta\\ &\qquad+\tfrac{1}{2}(3\mu_3-\mu_3'')\sin 5\eta\,;\\ \frac{2\widehat{\xi\eta}}{c^2h^4}&=[-2\mu_1'\sinh^2\xi+2\mu_3'+\mu_1\sinh 2\xi]\cos\eta\\ &\quad+3[\mu_3\sinh 2\xi-\mu_3'\cosh 2\xi]\cos 3\eta+\mu_3'\cos 5\eta, \end{aligned} \quad . \quad . \quad . \quad (17)$$

where

$$\mu_1=\frac{3p_0\sinh^2\alpha}{2}\left[-2e^{\xi}+e^{2\alpha-\xi}+e^{2\alpha-3\xi}+2\sinh^3\xi\,\mathrm{cosech}^2\alpha\right]$$

and

$$\begin{aligned}\mu_3=\frac{p_0\sinh^2\alpha}{2}\Big[&3e^{2\alpha-\xi}-e^{3\alpha-3\xi}(3\cosh\alpha+\sinh\alpha)\\ &-2\sinh^3\xi\,\mathrm{cosech}^2\alpha\Big].\end{aligned}$$

On the neutral axis $\widehat{\xi\xi}$ and $\widehat{\eta\eta}$ vanish, while $\xi\eta$ is not zero on this axis except at the vertices of the ellipse.

On the boundary of the elliptic hole $\xi=\alpha$,

$$\begin{aligned}\widehat{\eta\eta}=&\frac{12p_0\sinh\alpha}{c^2(\cosh 2\alpha-\cos 2\eta)}(\cosh\alpha+\sinh\alpha)\\ &\times[(3\cosh\alpha-\sinh\alpha)\sin\eta-(\cosh\alpha+\sinh\alpha)\sin 3\eta].\end{aligned} \quad (18)$$

It vanishes at the vertices of the ellipse, *i. e.*, when $\eta = n\pi$ and also when η has values satisfying the relations

$$\sin^2\eta = \frac{\sinh\alpha}{\cosh\alpha + \sinh\alpha}.$$

At the ends of the minor axis

$$\widehat{\eta\eta} = \pm\frac{24p_0}{c^2}\sinh\alpha\left(1 + \frac{\sinh\alpha}{\cosh\alpha}\right)$$

$$= \pm \mathrm{A}b\left(1 + \frac{b}{a}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

a and b being the lengths of the semi-major axis and the semi-minor axis respectively. It can be easily seen that these values correspond to a pair of maximum and minimum values on the boundary. In the case of a circular hole $a = b$, and hence, when $\eta = \pm\frac{\pi}{2}$,

$$\widehat{\eta\eta} = \pm 2\mathrm{A}a.$$

These values tally with those obtained by Tuzi in the case of a circular hole.

When $\alpha = 0$, $\widehat{\eta\eta} = 0$ and the crack surface is free from any stress whatsoever.

§ 4. *The Major Axis of the Ellipse at Right Angles to the Neutral Axis of the Beam.*

Putting $\theta = \frac{\pi}{2}$, we get from (11)

$$\lambda_1 = \frac{3p_0\cosh^2\alpha}{2}\left[-2e^{\xi} + e^{2\alpha-\xi} - e^{2\alpha-3\xi} + 2\cosh^3\xi\operatorname{sech}^2\alpha\right],$$

$$\lambda_3 = \frac{p_0}{2}\cosh^2\alpha\left[-3e^{2\alpha-3\xi} + e^{3\alpha-3\xi}(\cosh\alpha + 3\sinh\alpha) + 2\cosh^3\xi\operatorname{sech}^2\alpha\right],$$

$$\mu_1 = \mu_3 = 0.$$

The values of stresses can now be deduced from (12).

On the elliptic boundary $\xi = \alpha$,

$$\widehat{\eta\eta} = \frac{12p_0(\cosh\alpha + \sinh\alpha)}{c^2(\cosh 2\alpha - \cos 2\eta)}\left[(3\sinh\alpha - \cosh\alpha)\cosh\alpha\cos\eta + (\cosh\alpha + \sinh\alpha)\cosh\alpha\cos 3\eta\right] \quad . \quad . \quad . \quad (20)$$

When $\eta = (2n+1)\frac{\pi}{2}$, $\widehat{\eta\eta} = 0$, *i. e.*, at the ends of the minor

axis, which is along the neutral axis in the present case, there is no stress. On the elliptic boundary $\widehat{\eta\eta}$ is also zero for values of η which satisfy the relation

$$\cos^2\eta = \frac{\cosh\alpha}{\cosh c + \sinh\alpha}.$$

Values of $\widehat{\eta\eta}$ at the ends of the major axis are

$$\pm \mathrm{A}a\left(1+\frac{a}{b}\right). \quad . \quad . \quad . \quad . \quad . \quad (21)$$

These are the numerically greatest values of $\widehat{\eta\eta}$ on the boundary.

When $\alpha=0$ we get

$$\widehat{\eta\eta} = -\mathrm{A}c\cos\eta, \qquad . \quad . \quad . \quad . \quad (22)$$

which shows that this stress has got the greatest values at the ends of the crack and zero values at its centre, and is negative when $\frac{\pi}{2} > \eta > -\frac{\pi}{2}$, and positive when $\frac{3\pi}{2} > \eta > \frac{\pi}{2}$.

§ 5. *Conclusion.*

From what has been considered here, it is found that the values of the stress $\widehat{\eta\eta}$ acting close to the free surface of a small crack and directed along it are different in the three different cases:

(1) Close to the free surface of the crack, $\widehat{\eta\eta}$ vanishes if the crack lies along the neutral axis of the bent beam.

(2) $\widehat{\eta\eta}$ has a value $-\mathrm{A}c\cos\eta$ at any point defined by η close to the free surface when the crack is at right angles to the neutral axis of the bent beam, the greatest numerical values being obtained at the extremities of the crack.

(3) In the case of the crack lying at an inclination θ to the neutral axis, the value of $\widehat{\eta\eta}$ at any point close to the free surface of the crack depends on θ and the coordinates of the point, and it assumes infinitely large values at the extremities of the crack.

When the centre of the hole is not on the neutral axis, but at a height κ above it, the stresses can be similarly determined by writing down for χ_0 in (7) a value equal to $\frac{\mathrm{A}}{6}(x'\sin\theta+y'\cos\theta-\kappa)^3$ and introducing consequent modifications in the subsequent results.

Islamia College, Calcutta.
January 1931.

XXVI. *The Vibrations of Membranes and Plates.* By
ROBERT CAMERON COLWELL, *Physics Department, West Virginia University* *.

THE differential equation of a stretched membrane is

$$\frac{\partial^2 w}{\partial t^2} = c^2 \left\{ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right\}. \quad . \quad . \quad . \quad . \quad (1)$$

If the edges are fixed and the membrane a square with sides of length a, then $w=0$ for $x=a$, $x=0$ and $y=a$, $y=0$ provided the origin is taken at one corner of the square. The appropriate general solution is given by Rayleigh † in the form:

$$w = \sum_{m=1}^{m=\infty} \sum_{n=1}^{n=\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{a} \{ A_{mn} \cos pt + B_{mn} \sin pt \}. \quad (2)$$

The nodal lines of such a membrane can all be determined from equation (2).

If, however, we regard the stretched membrane as so supported that every point of the circumference is free to move along lines perpendicular to the plane of the membrane, we have a vibrating body which resembles in some ways a vibrating plate. In this paper, the vibrations of such a membrane will be compared with those of a Chladni plate supported either at the centre (the usual method) or at the edges. Since the required solutions of equation (1) are all sine and cosine functions, any figure obtained for a plate with side a may be reflected in these sides, which are always nodes or loops, and thus another figure obtained for a plate of sides $2a$: this reflexion may be continued indefinitely.

In fig. 1 the square membrane AOBC is so supported that the edges are free to vibrate perpendicularly to the plane of the membrane; the conditions to be satisfied are

$$\left.\begin{aligned} \frac{\partial w}{\partial x} &= 0 \text{ for } x = 0 \text{ and } x = a. \\ \frac{\partial w}{\partial y} &= 0 \text{ for } y = 0 \text{ and } y = a. \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

* Communicated by the Author.

† Rayleigh, 'Sound,' vol. i. chap. ix. p. 307 *et seq.*

The solution of equation (1) in periodic motion, so far as the nodal lines are concerned, is then

$$w \propto \left(\mathrm{A}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{a}+\mathrm{B}\cos\frac{n\pi x}{a}\cos\frac{m\pi y}{a}\right). \quad . \qquad (4)$$

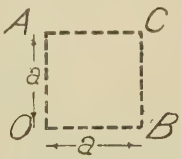
For $m=n=2$, and $\mathrm{A}=\mathrm{B}$, we have

$$\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}=0. \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

From this equation,

$$x=a/4,\ 3a/4 \quad \text{and} \quad y=a/4,\ 3a/4.$$

Fig. 1.



The nodal lines are shown in fig. 2. A reflexion of this figure in AO and OB will give fig. 3.

Fig. 2.

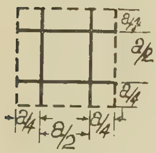
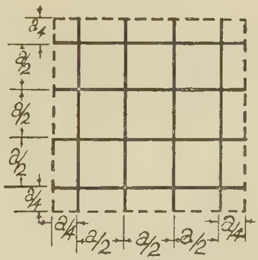


Fig. 3.



For $m=n=3$ and $\mathrm{A}=\mathrm{B}$ in equation (4),

$$x=a/6,\ a/2,\ 5a/6 \quad \text{and} \quad y=a/6,\ a/2,\ 5a/6.$$

Thus all the straight lines parallel to the sides may be drawn for a Chladni plate from equation (4) when $m=n$ and $\mathrm{A}=\mathrm{B}$. All these figures will give an internode at the centre.

For $n=0$ and $\mathrm{A}=\mathrm{B}$ in equation (4),

$$\cos\frac{m\pi x}{a}+\cos\frac{m\pi y}{a}=0. \quad . \quad . \quad . \quad . \qquad (6)$$

If $m=2$,

$$\cos\frac{\pi(x+y)}{a}\cos\frac{\pi(x-y)}{a}=0, \quad . \quad . \quad . \quad . \quad (7)$$

or

$$(x+y)=a/2,\ 3a/2 \quad \text{and} \quad x-y=\pm a/2.$$

These equations represent the inscribed square of fig. 4; and fig. 5 by reflexion.

Fig. 4.

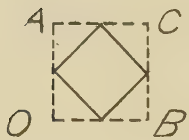
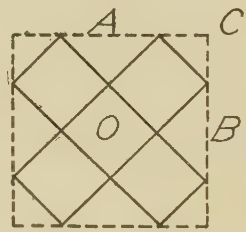


Fig. 5.



For $n=0$, $m=2$ and $\mathrm{A}=-\mathrm{B}$, equation (4) becomes

$$\cos\frac{2\pi x}{a}-\cos\frac{2\pi y}{a}=0, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\sin\frac{\pi(x+y)}{a}\sin\frac{\pi(x-y)}{a}=0.. \quad . \quad . \quad . \quad (9)$$

These equations give the two diagonals of fig. 6, which is directly applicable to a Chladni plate fixed at the centre. The reflexion gives fig. 7.

Fig. 6.

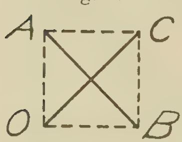
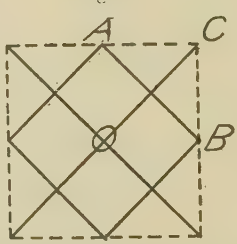


Fig. 7.



In all these diagrams the nodal lines are continuous, while the internodes (loops) are dotted.

For $m=2$, $n=1$, and $\mathrm{A}=-\mathrm{B}$, equation (4) becomes

$$\cos\frac{2\pi x}{a}\cos\frac{\pi y}{a}=\cos\frac{\pi x}{a}\cos\frac{2\pi y}{a}.. \quad . \quad . \quad (10)$$

This reduces to

$$2\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\left(\cos\frac{\pi x}{a}-\cos\frac{\pi y}{a}\right)=\cos\frac{\pi y}{a}-\cos\frac{\pi x}{a}, \quad (11)$$

so that

$$x=y \quad \text{and} \quad \cos\frac{\pi x}{a}\cos\frac{\pi y}{a}=-\tfrac{1}{2}. \quad . \quad . \quad (12)$$

Equation (12) is plotted in fig. 8 and two reflexions in

Fig. 8.

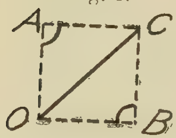
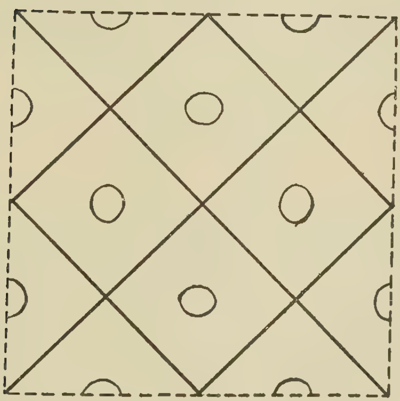


Fig. 9.

Fig. 10.



figs. 9 and 10. They also give approximate forms for a Chladni plate.

For $m=3$, $n=1$, equation (4) becomes

$$A\cos\frac{3\pi x}{a}\cos\frac{\pi y}{a}+B\cos\frac{\pi x}{a}\cos\frac{3\pi y}{a}=0, \quad . \quad (13)$$

which reduces to

$$\cos\frac{\pi x}{a}\cos\frac{\pi y}{a}\left[(A+B)-4A\sin^2\frac{\pi x}{a}-4B\sin^2\frac{\pi y}{a}\right]=0.$$

If A=B, the equation is a circle and two straight lines,

$$\sin^2\frac{\pi x}{a}+\sin^2\frac{\pi y}{a}=\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad (14)$$

This equation with one reflexion is plotted in figs. 11 and 12.

A Chladni plate is usually fixed at the centre, so that many of the nodal curves pass through that point. In order to determine the shape of such curves, it is necessary to investigate a membrane with two fixed edges and two free edges.

Fig. 11.

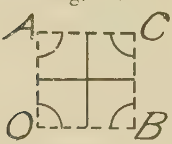
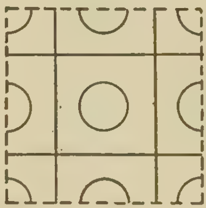
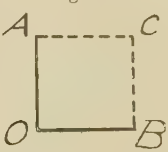


Fig. 12.



In fig. 13, AO and OB are fixed, while AC and BC are free; the boundary conditions are $w=0$ for $x=0$ and $y=0$

Fig. 13.



and $\frac{\partial w}{\partial x}=0$, $\frac{\partial w}{\partial y}=0$ for $x=a$ and $y=a$. The nodal lines are therefore determined by

$$\mathrm{A}\sin\frac{m\pi x}{2a}\sin\frac{n\pi y}{2a}+\mathrm{B}\sin\frac{n\pi x}{2a}\sin\frac{m\pi y}{2a}=0, \quad . \quad (15)$$

where m and n are odd numbers only. In this equation $m=n$, A=B will give the straight line figures similar to figs. 2 and 3, but the centre point will always be in a nodal line.

For $m=3$, $n=1$, A=B, equation (15) reduces to

$$\sin^2\left(\frac{\pi x}{2a}\right)+\sin^2\left(\frac{\pi y}{2a}\right)=\frac{3}{2}. \quad . \quad . \quad . \quad . \quad (16)$$

This equation with one reflexion is shown in figs. 14 and 15.

More complicated figures may be developed by using higher values for m and n and different ratios for A and B.

Chladni plates are usually set in vibration with a violin bow; but I have found that a valve oscillator with a suitable mechanical vibrator at the output will set a plate in vibration

Fig. 14.

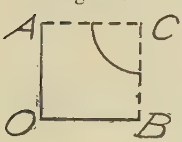
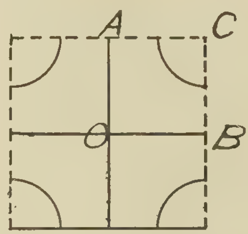
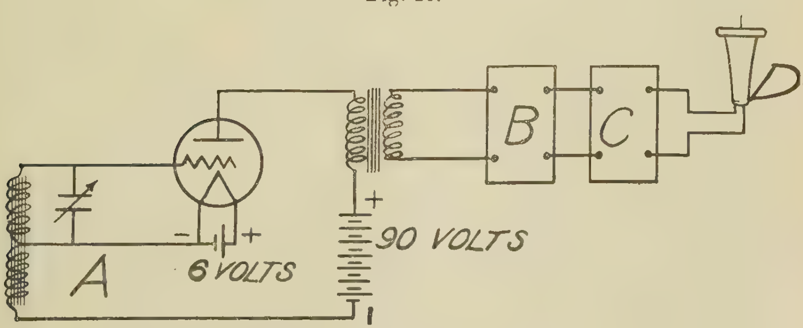


Fig. 15.



over a very wide range of values, and so give many symmetrical figures which have hitherto escaped notice. The valve oscillator shown in fig. 16 is of conventional design; the output from the valve is passed through two stages of resistance-coupled amplifiers and then to a commercial amplifier such as is used for public addresses. The output from

Fig. 16.

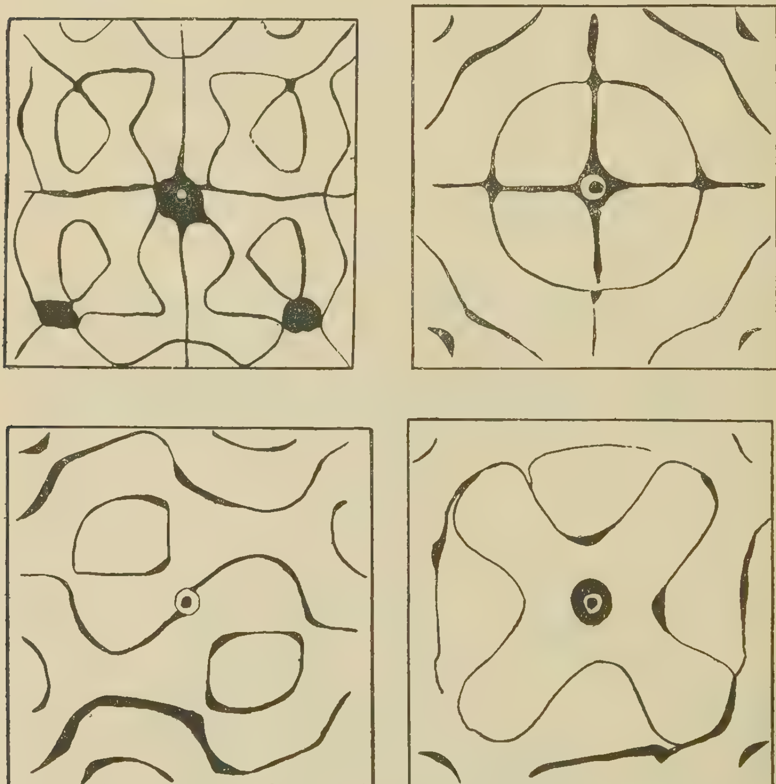


A. Audio oscillator—variable.
B. Two-stage amplifier, resistance coupled.
C. Two-stage commercial amplifier, transformer coupled.
D. Telephone receiver with nail.

the last amplifier passes through an ordinary telephone receiver. To the centre of the diaphragm of this receiver is soldered a small nail. The point of this nail is pressed against the underside of the Chladni plate. The pitch of

the note is then varied until the plate is set in violent vibration and a sand figure formed.

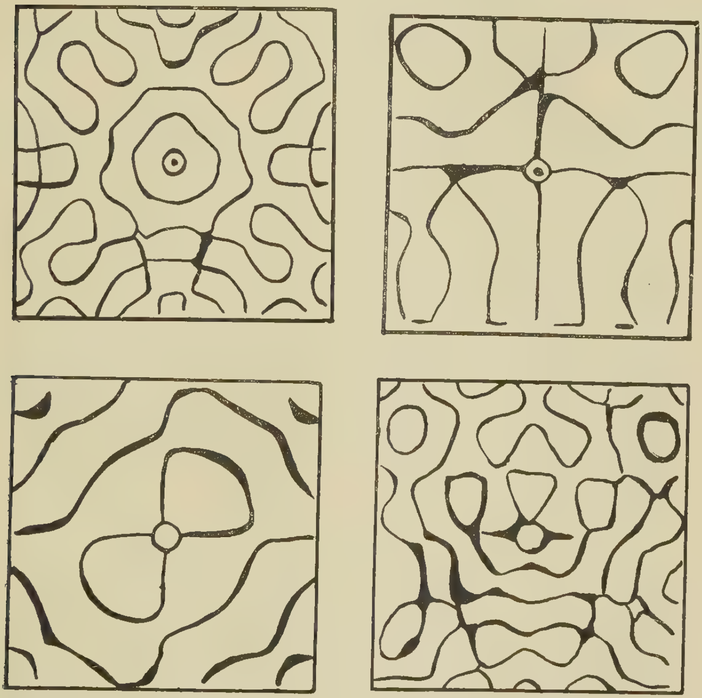
With this device it is possible to vibrate plates, membranes, and bells with both audible and supersonic vibrations; as yet, it has been applied mainly to Chladni plates. The vibrations of a Chladni plate which is entirely free have been



Group A.—Large brass plate fixed at centre. Bowing; valve oscillator.

arrived at in the following manner. Two parallel edges of the plate rest upon rubber mats overlapping the mats by one-half inch. The vibrator is applied until nodal lines appear faintly outlined along the edges. Matches are slipped under the plate beneath the nodal lines. Then the plate vibrates violently and the appropriate figure is formed. Four groups of figures are shown below according to the

method of formation. The large brass plate referred to is 12 in. square and 1/8 in. thick. The small brass plate is 10×10 in. square and 1/16 in. thick. Any departure from symmetry in these figures is due either to inhomogeneities in the plates, which are especially conspicuous at high frequencies, or to the fact that the vibrating nail has not been placod against the plate at exactly the right position—the mechanical coupling between the plate and

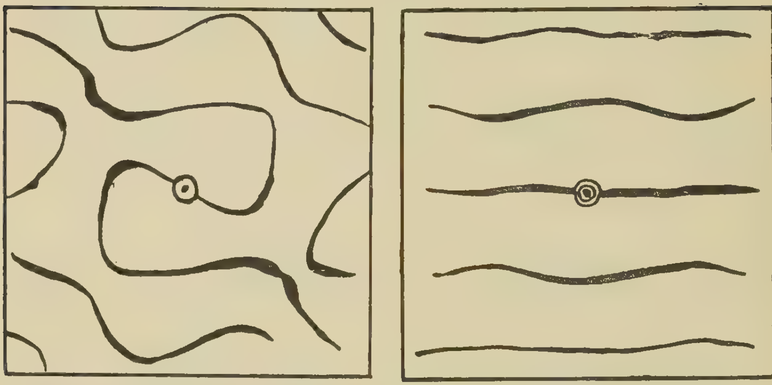


Group B.—Small brass plate fixed at centre. Valve oscillator.

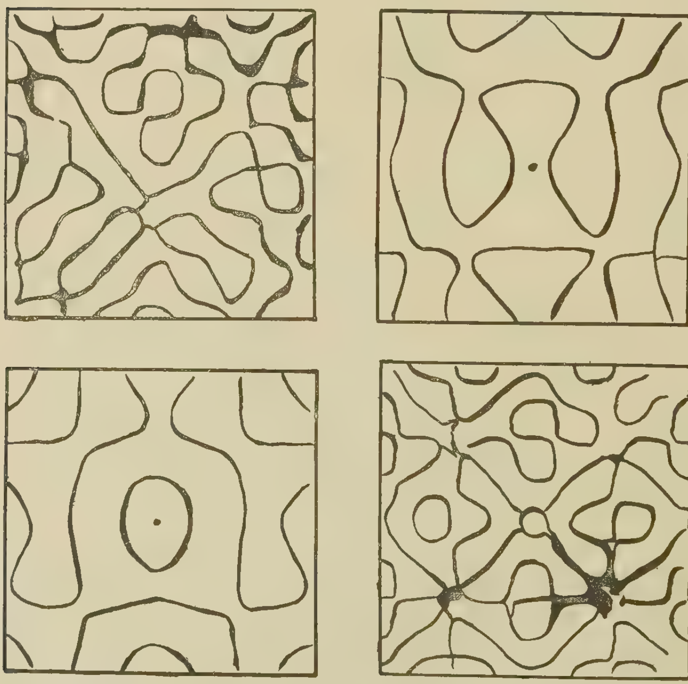
the nail is very critical. The figures from 1 to 15 have all been obtained with the vibrator, excepting 3, 4, 5, and 14.

From sixty figures made in this way, fourteen are shown here, arranged in four groups—A, B, C, and D. The caption of each group shows the conditions under which the sand figures were formed.

Some of the figures shown were obtained long ago with the violin bow, and are not new. This is the first time they have been developed with an electrical oscillator.



Group C.—Large brass plate resting on rubber mats. Valve oscillator.



Group D.—Small brass plate resting on rubber mats. Valve oscillator.

XXVII. *The Heat Loss from a Cylinder embedded in an Insulating Wall.* By F. H. Schofield, *B.A.*, *B.Sc.*, *Physics Department, National Physical Laboratory, Teddington, Middlesex* *.

Abstract.

The problem discussed is that of the steady flow of heat from an infinitely long circular cylinder with an isothermal boundary, embedded in an infinitely long wall with a rectangular isothermal boundary. For this two-dimensional problem the method of conformal representation is employed, taking the rectangular outer boundary and an imaginary inner isothermal boundary of such dimensions and shape that it yields a particular isothermal approximating, as closely as may be, with the required circle. Formulæ are derived for calculating the overall heat loss for cases where the axis of the cylinder lies either outside or in the mid-plane of the wall, and a complete distribution of isothermals and flow lines is calculated for two such cases. The distribution in these latter cases is compared with that given by a method of graphical estimation, and a determination of overall heat loss, based on electrical analogy, is given for one of the cases. An alternative method of theoretical treatment is also discussed.

I. Introduction.

This problem was brought to the author's notice in connexion with the calculation of the heat flow from a metallic cylinder maintained at certain temperature and surrounded by a material of very low conductivity contained within a vessel with thick copper walls. Under these conditions the inner and outer boundaries can safely be assumed to be isothermal. In other practical cases this assumption may not be justified. For example, in the case of an electric cable or a steam-pipe embedded in a wall, the surfaces of which have attained to temperature equilibrium under the influence of convection and radiation in a free atmosphere, the temperature of the outer boundary at a great distance from the cylinder will be that of the atmosphere, while at the part of the boundary nearest to the

* Communicated by the Author.

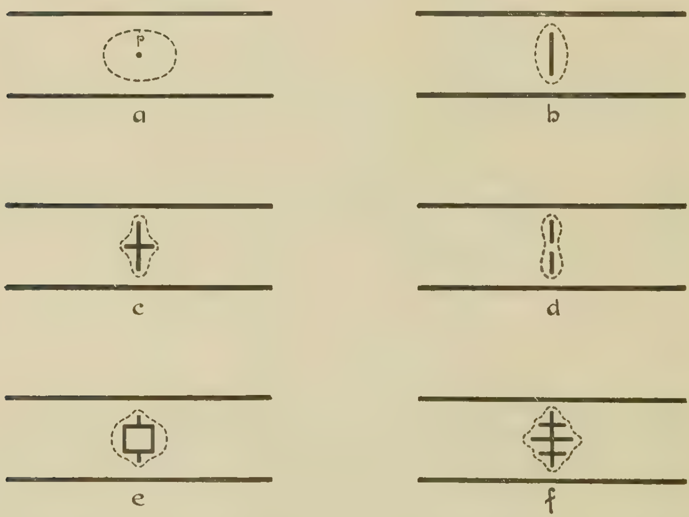
cylinder it will be considerably higher. Under these conditions the solution attempted below will not do more than give upper and lower limits to the heat loss according as the boundary is assumed to be isothermal at the minimum or maximum of temperature indicated.

While the problem has been stated above in terms of heat flow it could obviously be expressed in relation to any other phenomenon, *e. g.*, electrostatic, electrodynamic, hydrodynamic, which is subject to the Laplace equation.

II. Method of Treatment.

The conjugate function which gives an inner circular boundary and an outer rectangular boundary does not appear

Fig. 1.



to be known. In dealing with the matter below we take the outer rectangular boundary and an assumed inner boundary of such shape that it permits of treatment by the Christoffel-Schwarz method and gives an isothermal as nearly as possible coincident with the required circular isothermal. In fig. 1 are shown an infinitely wide wall and various shapes of inner boundary.

a is a point source which gives, as isothermals, ovals elongated horizontally. By taking the isothermal coincident with the required circle where it cuts the vertical axis

(*i. e.*, at p) an upper limit of the heat loss would obviously be obtained, while a lower limit would be given by the isothermal coincident with the circle where it cuts the horizontal axis.

b is a line source giving vertically elongated ovals for its nearest isothermals. Considering successive isothermals receding from the source there will obviously be one isothermal with equal intercepts on the vertical and horizontal axes, and this should give a fair approximation to a circle with a slight excess or deficiency between the axes. This case, unlike a, involves one parameter, namely, the length of the line source.

c and d are identical analytically, and each involves two parameters, so that, in addition to making the intercepts on the axes equal, an intermediate point of coincidence with the circle may be secured. A slight excess or deficiency given by b between the axes could thus be partially corrected by c or d as the case may be.

e and f are more complicated forms, involving respectively three or four parameters, and therefore giving a greater possibility of coincidence with the required circle.

The figures as drawn deal roughly with the case of a cylinder symmetrically placed in an infinitely wide wall, but arguments similar to the above would of course apply to a finite rectangular wall or to an asymmetrically placed cylinder. A detailed treatment of several of the cases is given below.

III. Cylinder Central: Source of Two or less Parameters.

We deal first with the case having two parameters (fig. 1, c), and, for sake of generality, take a finite rectangular wall. Inserting the lines of symmetry from the ends of the inner boundary, and adopting the convention of showing the boundary isothermals and flow lines as full and dashed lines respectively, we need consider only one quarter of the figure as shown in fig. 2 (z plane). Taking now for the z plane transformation that represented by the figures in brackets, which give the distances along the real axis in the intermediate (t) plane, the t and the final w planes are as represented in fig. 2 *. Here k, λ, and μ are each less than 1.

* For a treatment of other similar rectangular figures see papers by the author (Phil. Mag. vi. p. 567 (1928), and Phil. Mag. x. p. 480 (1930)). Also Moulton, Proc. Lond. Math. Soc. iii. p. 104 (1905).

The transformation from the z to the t plane is given by

$$dz/dt = \mathrm{P}t^{-1/2}(t-1/k^2)^{-1/2}(t+1/\mu^2)^{-1/2},$$

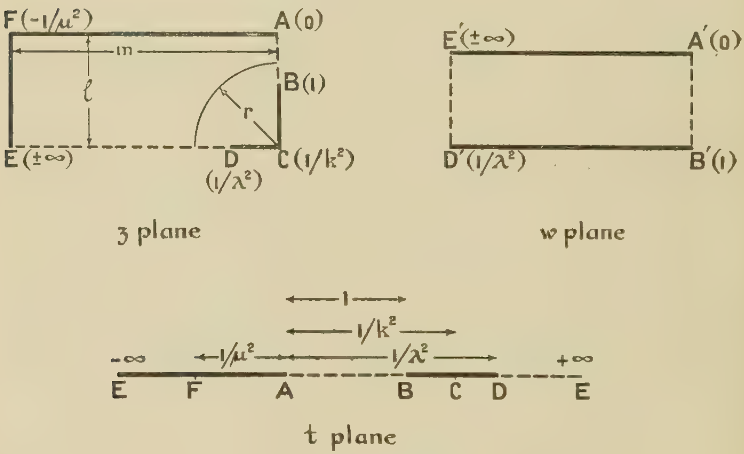
which, by the substitution $p^2=(1+\mu^2t)$, can be shown to yield an integration *

$$z = \mathrm{Q}\,\mathrm{sn}^{-1}\left[(1+\mu^2t)^{1/2},\ \{k^2/(k^2+\mu^2)\}^{1/2}\right]+\mathrm{R}.$$

For an origin at F and a real axis FA we have

$$\mathrm{R}=0, \qquad \mathrm{Q}=l/\mathrm{K}',$$

Fig. 2.



and hence

$$\sqrt{1+\mu^2t} = \mathrm{sn}\left\{z\mathrm{K}'/l,\ \sqrt{k^2/(k^2+\mu^2)}\right\}. \quad . \quad . \quad (1)$$

We shall find it convenient later on to have the origin at A with the real axis ABC, so that, putting $z = l\mathrm{K}/\mathrm{K}'+iz$,

$$\sqrt{1+\mu^2t} = \mathrm{sn}\left\{(\mathrm{K}+iz\mathrm{K}'/l),\ \sqrt{k^2/(k^2+\mu^2)}\right\}$$

$$= \frac{1}{\mathrm{dn}\left\{z\mathrm{K}/l,\ \sqrt{\mu^2/(k^2+\mu^2)}\right\}},$$

and hence

$$\sqrt{t} = \frac{1}{\sqrt{k^2+\mu^2}}\,\frac{\mathrm{sn}}{\mathrm{dn}}\left\{\frac{z\mathrm{K}}{l},\ \left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\},$$

* Since the function is to be inverted it is here written sn^{-1} in place of Legendre's F.

the modulus being known in terms of the linear dimensions l and m, since

$$\frac{m}{l}=\frac{\mathrm{K}'}{\mathrm{K}}\operatorname{mod}\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}. \quad . \quad . \quad . \quad . \quad (2)$$

Turning now to the transformation from the w to the t plane, we have

$$dw/dt=\mathrm{S}t^{-1/2}(t-1)^{-1/2}(t-1/\lambda^2)^{-1/2},$$

which yields on integration for an origin A′ and real axis A′B′

$$w=\mathrm{B}\operatorname{sn}^{-1}(\sqrt{t},\lambda), \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where B is a constant which need not be evaluated, since it determines merely the unit in which w is expressed.

Eliminating t between (1) and (3),

$$\frac{\operatorname{sn}}{\operatorname{dn}}\left\{\frac{z\mathrm{K}}{l},\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}=(k^2+\mu^2)^{1/2}\operatorname{sn}\left\{\frac{w}{\mathrm{B}},\lambda\right\}. \quad (4)$$

Here z and w are complex variables, which we may call $(x+iy)$ and $(u+iv)$ respectively.

Now equation (4) gives the distribution of isothermals and flow lines in z plane in terms of the rectilinear distribution in the w plane, and since in the latter plane temperatures are measured along the real axis we may refer to the variable u as temperature and similarly v as heat flux. Then for the temperature distribution along AB we have, putting $y=0$ and $v=0$,

$$\frac{\operatorname{sn}}{\operatorname{dn}}\left\{\frac{x\mathrm{K}}{l},\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}=(k^2+\mu^2)^{1/2}\operatorname{sn}\left\{\frac{u}{\mathrm{B}},\lambda\right\}.$$

Hence at the point where the circle of radius r (fig. 2) meets AB we have the temperature u given by

$$\operatorname{sn}\left\{\frac{u}{\mathrm{B}},\lambda\right\}=\frac{1}{(k^2+\mu^2)^{1/2}}\frac{\operatorname{sn}}{\operatorname{dn}}\left\{\mathrm{K}\left(\frac{l-r}{l}\right),\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}$$

$$=\frac{1}{k}\operatorname{cn}\left\{\frac{r\mathrm{K}}{l},\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}. \quad . \quad . \quad . \quad . \quad (5)$$

For the temperature distribution along DE we have from (4), putting

$$z=l+iy \quad \text{and} \quad w/\mathrm{B}=i\mathrm{K}'+u\mathrm{B},$$

$$(k^2+\mu^2)^{1/2}\operatorname{sn}\{(i\mathrm{K}'+u/\mathrm{B}),\lambda\}$$

$$=\frac{\operatorname{sn}}{\operatorname{dn}}\left\{\left(\mathrm{K}+\frac{iy\mathrm{K}}{l}\right),\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}$$

$$\therefore \quad \frac{1}{\lambda\,\mathrm{sn}\,(u/\mathrm{B},\lambda)} = \frac{1}{k\,\mathrm{cn}\left\{\frac{y\mathrm{K}'}{l},\left(\frac{k^2}{k^2+\mu^2}\right)^{1/2}\right\}} \quad . \quad . \quad . \quad (6)$$

And since $y=r$ where the circle meets CE we have, on multiplying (5) and (6),

$$\frac{\lambda}{k^2} = \frac{\mathrm{cn}\left\{\frac{r\mathrm{K}'}{l},\left(\frac{k^2}{k^2+\mu^2}\right)^{1/2}\right\}}{\mathrm{cn}\left\{\frac{r\mathrm{K}}{l},\left(\frac{\mu^2}{k^2+\mu^2}\right)^{1/2}\right\}} \quad . \quad . \quad . \quad . \quad . \quad (7)$$

As already pointed out, the modulus and its complement on the right-hand side of the equation are known in terms of the linear dimensions l and m through equation (2). The parameters λ and k are connected by the equation (7) by virtue of the fact that coincidence with the circle has been obtained where it cuts the axes. By trial with various combinations of λ and k in equation (4) an intermediate point of coincidence could also be obtained, and this would fix the two parameters. When this has been done the "shape factor" (S)*, *i.e.*, the ratio of the total flux to the temperature difference, is obtained as follows :—

From (3) the total flux for the complete circle is given by

$$v = 4\mathrm{BK}'\,(\mathrm{mod}\,\lambda);$$

from (5)

temperature difference (u)

$$= \mathrm{B}\,\mathrm{sn}^{-1}\left\{\frac{\mathrm{cn}\,(r\mathrm{K}/l,\ \sqrt{\mu^2/(k^2+\mu^2)}}{k}\right\}\ \mathrm{mod}\,\lambda.$$

Hence

$$\mathrm{S} = \frac{4\mathrm{K}'}{\mathrm{sn}^{-1}\left\{\frac{\mathrm{cn}\,(r\mathrm{K}/l,\ \sqrt{\mu^2/(k^2+\mu^2)})}{k}\right\}}\,(\mathrm{mod}\,\lambda). \quad . \quad (8)$$

The successive stages in the simplification of this solution down to the case of a point source in an infinite wall (fig. 1, a) are as follows :—

Infinite Wall with Two Parameter Source (fig. 1, c or d).

The wall is made infinitely wide by putting $\mu=0$ (see fig. 2). Equation (4), giving the general distribution, now reads

$$\sin\,(z\pi/2l) = k\,\mathrm{sn}\,(w/\mathrm{B},\lambda). \quad . \quad . \quad . \quad . \quad (9)$$

* The quantity $\mathrm{S}/4\pi$ would be the electrical capacity in the corresponding electrostatic problem.

Equation (7), giving the relation of the parameters, becomes

$$k^2/\lambda = \cos(r\pi/2l)\cosh(r\pi/2l), \quad . \quad . \quad . \quad (10)$$

while the shape factor is given by

$$S = \frac{4K'}{\mathrm{sn}^{-1}\left\{\dfrac{\cos r\pi/2l}{k}\right\}} (\mathrm{mod}\,\lambda). \quad . \quad . \quad . \quad (11)$$

We note from (10) that k may be either greater or less than λ, corresponding with fig. 1, c or d.

Infinite Wall with Line, or One Parameter, Source (fig. 1, b).

This is obtained by putting $\lambda = k$, whence

$$k = \cos(r\pi/2l)\cosh(r\pi/2l), \quad . \quad . \quad . \quad (12)$$

and

$$S = \frac{4K'}{\mathrm{sn}^{-1}(\mathrm{sech}\, r\pi/2l)} (\mathrm{mod}\, k). \quad . \quad . \quad . \quad (13)$$

Infinite Wall with Point Source (fig. 1, a).

This is obtained by putting $k=1$, so that the general equation becomes

$$\sin(z\pi/2l) = \tanh(w/B). \quad . \quad . \quad . \quad . \quad (14)$$

Calling S_U the shape factor when coincidence of the isothermal with the circle is on the vertical axis, we have

$$S_U = \frac{2\pi}{\tanh^{-1}(\cos r\pi/2l)} = \frac{2\pi}{\log\cot(r\pi/4l)}. \quad . \quad (15)$$

The distribution along the horizontal axis CE (see fig. 2) is given by

$$\cosh(y\pi/2l) = \coth(u/B), \quad . \quad . \quad . \quad (16)$$

and hence S_L, the shape factor when coincidence of the isothermal with the circle of radius r is on the horizontal axis, is given by

$$S_L = \frac{2\pi}{\coth^{-1}(\cosh r\pi/2l)} = \frac{2\pi}{\log\coth(r\pi/4l)}. \quad . \quad (17)$$

S_U and S_L give, of course, upper and lower limits for the required heat loss.

IV. Cylinder Central in Infinite Wall: further Treatment with Line Source.

The calculation of the shape factor in this case is covered by equations (12) and (13), which, as already explained, relate to an isothermal made to coincide with the required circle where it cuts the two axes. We propose to consider the departure from the circle of this isothermal at some intermediate point, and hence obtain an idea of the closeness of the approximation.

If we transfer the origin in the z plane to C (fig. 2), the general equation * becomes

$$\cos(z\pi/2l) = k\,\mathrm{sn}(w/\mathrm{B}, k), \quad . \quad . \quad . \quad (18)$$

where, as already pointed out, B is an arbitrary constant and

$$k = \cos(r\pi/2l)\cosh(r\pi/2l).$$

Splitting (18) into real and imaginary parts, and writing X for $(x\pi/2l)$, Y for $(y\pi/2l)$, R for $(r\pi/2l)$, U for u/B, V for v/B, we have

$$\cos \mathrm{X} \cosh \mathrm{Y} = \frac{k\,\mathrm{sn}(\mathrm{U}, k)\,\mathrm{dn}(\mathrm{V}, k')}{\mathrm{cn}^2(\mathrm{V}, k') + k^2\,\mathrm{sn}^2(\mathrm{U}, k)\,\mathrm{sn}^2(\mathrm{V}, k')}, \quad (19)$$

$$\sin \mathrm{X} \sinh \mathrm{Y} = -\frac{k\,\mathrm{cn}(\mathrm{U}, k)\,\mathrm{dn}(\mathrm{U}, k)\,\mathrm{sn}(\mathrm{V}, k')\,\mathrm{cn}(\mathrm{V}, k')}{\mathrm{cn}^2(\mathrm{V}, k') + k^2\,\mathrm{sn}^2(\mathrm{U}, k)\,\mathrm{sn}^2(\mathrm{V}, k')}. \quad . \quad . \quad . \quad (20)$$

Now U is the temperature along the x axis, where $x=r$, and is therefore given by (18) as follows:—

$$\cos \mathrm{R} = k\,\mathrm{sn}(\mathrm{U}, k); \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and hence

$$\mathrm{sn}(\mathrm{U}, k) = \mathrm{sech}\,\mathrm{R},$$

$$\mathrm{cn}(\mathrm{U}, k) = \tanh \mathrm{R},$$

$$\mathrm{dn}(\mathrm{U}, k) = \sin \mathrm{R}.$$

Now V varies from 0 to (K, k'), and if we take the medial flow line $\mathrm{V}=(\mathrm{K}/2, k')$, we have

$$\mathrm{sn}(\mathrm{K}/2, k') = [1/(1+k)]^{1/2},$$

$$\mathrm{cn}(\mathrm{K}/2, k') = [k/(1+k)]^{1/2},$$

$$\mathrm{dn}(\mathrm{K}/2, k') = (k)^{1/2}.$$

* *Cf.* equation (9).

Hence for the medial flow line

$$\cos X \cosh Y = \frac{\cos R(1+k)\sqrt{k}}{k+\cos^2 R} = \frac{(1+k)\sqrt{k}}{\cosh R+\cos R} = P, \quad . \quad . \quad . \quad (22)$$

$$\sin X \sinh Y = -\frac{\cos R \sin R \sinh R \sqrt{k}}{k+\cos^2 R}$$

$$= -\frac{\sin R \sinh R \sqrt{k}}{\cosh R+\cos R} = Q; \quad (23)$$

from which, by elimination of X and Y respectively,

$$\cosh 2Y = P^2+Q^2+[(P^2+Q^2)^2+1-2(P^2-Q^2)]^{1/2}, \quad . \quad (24)$$

$$\cos 2X = (P^2+Q^2)-[(P^2+Q^2)^2+1-2(P^2-Q^2)]^{1/2}. \quad (25)$$

For dealing with these expressions we have

$$P^2+Q^2 = \cosh R \cos R, \quad . \quad . \quad . \quad . \quad . \quad (26)$$

$$[(P^2+Q^2)^2+1-2(P^2-Q^2)]^{1/2}$$

$$= (1+\cosh R \cos R)\left\{\frac{\cosh R-\cos R}{\cosh R+\cos R}\right\}, \quad (27)$$

and hence, expanding in powers of R up to R^6, and substituting in (24) and (25),

$$\cosh 2Y = 1+R^2-\frac{R^4}{6}-\frac{11R^6}{90}, \quad . \quad . \quad . \quad (28)$$

$$\cos 2X = 1-R^2-\frac{R^4}{6}+\frac{11R^6}{90}. \quad . \quad . \quad . \quad (29)$$

From which may be derived

$$Y^2 = \frac{R^2}{2}\left\{1-\frac{R^2}{3}-\frac{R^4}{45}\right\}, \quad . \quad . \quad . \quad . \quad (30)$$

$$X^2 = \frac{R^2}{2}\left\{1+\frac{R^2}{3}-\frac{R^2}{45}\right\}, \quad . \quad . \quad . \quad . \quad (31)$$

and therefore

$$X^2+Y^2 = R^2(1-R^4/45). \quad . \quad . \quad . \quad . \quad (32)$$

So that, finally, we have for the medial flow line

$$(x^2+y^2)_m{}^{1/2} = r\left\{1-\frac{r^4\pi^4}{1440l^4}\right\}. \quad . \quad . \quad . \quad (33)$$

It follows from (33) that the isothermal where it cuts the medial flow line comes slightly within the required circle.

It is reasonable to suppose that the departure from the circle would be about a maximum in this position, as is indeed indicated by a detailed calculation for a particular case shown on fig. 5. Consequently the shape factor given by equations (12) and (13) will be a lower limit. We can, however, readily obtain an upper limit. Thus from (33) a value of r' can be found which will make $(x^2+y^2)^{1/2}$ equal to the required r, *i. e.*, will make a line source isothermal coincident with the circle where it cuts the medial flow line. This value of r' will clearly give an upper limit when substituted in (12) and (13).

For the purpose of illustrating the use of point and line sources we give in Table I. below numerical examples for a range of values of r/l. Thus columns 1 and 2 indicate the departure from the circle of the line source isothermal where it cuts the medial flow line; columns 3 and 4 give the upper

TABLE I.

	Line source.			Point source.			
r/l.	$[(x^2+y^2)/l^2]^{\frac{1}{2}}$ from (33).	S_U.	S_L.	S_U.	S_L.	$\frac{1}{2}(S_U+S_L)$.	$(S_U \times S_L)^{\frac{1}{2}}$.
0·8	0·778	15·51	14·28	19·75	10·73	15·24	14·56
0·6	0·595	8·54	8·43	9·32	7·63	8·47	8·43
0·4	0·399	5·44	5·43	5·59	5·28	5·43	5·43
0·2	0·200	3·39	3·39	3·42	3·37	3·39	3·39

and lower limits of the shape factor for a line source; while the remaining columns give the corresponding limits for a point source, together with their arithmetic and geometric means. It will be seen that the departure from the circle is small except in the case of relatively large circles, that the line source fixes the shape factor within narrow limits, and that, though the point source gives wider limits, its mean values are not far from the true shape factor.

The closeness with which the isothermal, derived from a line source, can be made to imitate a circle renders it hardly profitable to pursue the higher degrees of approximation represented by more complicated sources such as these in fig. 1, *c*, *d*, *e*, or *f*. However, *c* and *d* have been partly treated above, and it may be mentioned that the *z* plane transformations applicable to fig. 1, *e* and *f* have previously been dealt with by the author* in another connexion.

* Phil. Mag. vi. pp. 578, 582, 592 (1928).

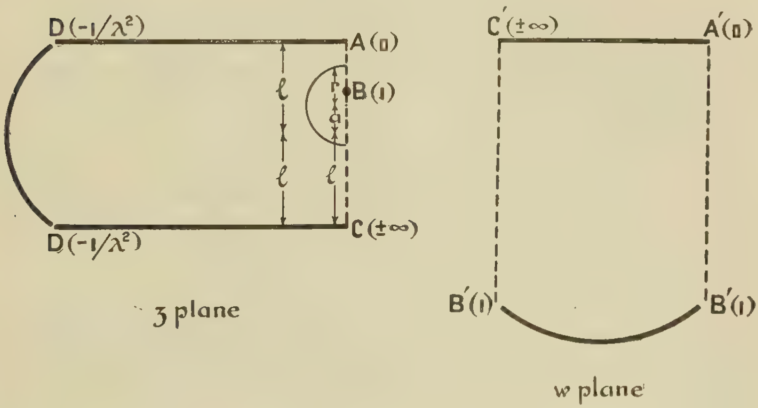
V. Cylinder Eccentric in Infinite Wall.

So far we have dealt with a cylinder having its axis in the mid-plane of the wall, and it is now proposed to consider the case where the axis is displaced parallel to the mid-plane. The treatment is similar to that already adopted, both point and line sources being taken for the inner boundry.

Point Source—Upper Limit.

The point source involves one parameter, namely, that required to fix its position, and it may be dealt with by means of the transformation indicated in fig. 3, which shows the z

Fig. 3.



and w planes only. As before, the boundary isothermals and flow lines are indicated by full and dashed lines respectively, the whole or portion of either at infinity being represented by a curved line. Thus the point isothermal B in the z plane which becomes the side of the rectangle at infinity in the w plane is there indicated as the full curved line B′B′, while the isothermal ADDC in the z plane has the curved portion DD to indicate that it extends to infinity and back.

By integration of the transformation equations and elimination of the intermediate variable t it will be found that the general relation of the z and w planes is

$$\tan(z\pi/4l) = \lambda \tanh(w/\mathrm{B}), \quad . \quad . \quad . \quad . \quad (34)$$

where λ and B are constants of which the latter need not be determined.

For the upper limit we have the condition that an isothermal must pass through the points $x=(l-a-r)$ and $x=(l-a+r)$ on the real axis. This gives

$$\lambda \tanh \mathrm{U} = \tan\left[\frac{\pi}{4}-\frac{(r+a)\pi}{4l}\right], \quad . \quad . \quad . \quad (35)$$

$$\lambda \coth \mathrm{U} = \tan\left[\frac{\pi}{4}+\frac{(r-a)\pi}{4l}\right]. \quad . \quad . \quad . \quad (36)$$

Hence by eliminating λ

$$\tanh^2 \mathrm{U} = \tan\left[\frac{\pi}{4}-\frac{(r+a)\pi}{4l}\right] \div \tan\left[\frac{\pi}{4}+\frac{(r-a)\pi}{4l}\right]$$

$$=\frac{\cos(a\pi/2l)-\sin(r\pi/2l)}{\cos(a\pi/2l)+\sin(r\pi/2l)}, \quad . \quad . \quad . \quad . \quad . \quad (37)$$

and since the total flux for the complete cylinder is $\pi\mathrm{B}$ instead of $2\pi\mathrm{B}$ as before, the upper limit of the shape factor is given by

$$\mathrm{S_U} = \frac{\pi}{\tanh^{-1}\left[\frac{\cos(a\pi/2l)-\sin(r\pi/2l)}{\cos(a\pi/2l)+\sin(r\pi/2l)}\right]^{1/2}}. \quad . \quad (38)$$

Point Source—Lower Limit.

For a lower limit we have the condition that the isothermal touches the circle at the end of its horizontal diameter, *i. e.*, at the point whose coordinates are $(l-a)$ and r. At this point therefore

$$\left[\frac{\partial y}{\partial x}\right]_{\mathrm{U=constant}} = 0.$$

By partial differentiation of (34) with respect to v we have

$$\frac{\pi}{4l}\sec^2\left(\frac{z\pi}{4l}\right)\left\{\frac{\partial x}{\partial v}+i\frac{\partial y}{\partial v}\right\} = i\frac{\lambda}{\mathrm{B}}\operatorname{sech}^2\frac{w}{\mathrm{B}};$$

$$\therefore \quad \frac{\partial x}{\partial v}+i\frac{\partial y}{\partial v} = i\mathrm{D}\{\lambda^2\cos^2(z\pi/4l)-\sin^2(z\pi/4l)\}$$

$$=\frac{i\mathrm{D}}{2}\{\lambda^2[1+\cos(z\pi/2l)]-1+\cos(z\pi/2l)\},$$

$$. \quad . \quad . \quad (39)$$

where D is a constant. In order to fulfil the condition set

out above, the imaginary part of the right-hand side of this equation must be equal to zero for the point $[(l-a), r]$, so that

$$\lambda^2 = \frac{1 - \sin(a\pi/2l)\cosh(r\pi/2l)}{1 + \sin(a\pi/2l)\cosh(r\pi/2l)}. \quad . \quad . \quad (40)$$

Hence to find the temperature u of the required isothermal we substitute in (34) the above value for λ, $(l-a)$ for x and r for y, and split the equation into real and imaginary parts. After some reduction this process yields

$$\tanh 2U = \frac{[1 - \sin^2(a\pi/2l)\cosh^2(r\pi/2l)]^{1/2}}{\cos(a\pi/2l)\cosh(r\pi/2l)}, \quad . \quad (41)$$

and hence the lower limit of the shape factor is given by

$$S_L = \frac{2\pi}{\tanh^{-1}\left[\dfrac{\{1 - \sin^2(a\pi/2l)\cosh^2(r\pi/2l\}^{1/2}}{\cos(a\pi/2l)\cosh(r\pi/2l)}\right]}. \quad . \quad (42)$$

It is useful to check the formulæ (38) and (42) by degenerating them to give the marginal cases of a cylinder centrally placed in a wall of finite thickness and one near the surface of a semi-infinite wall. The former is, of course, obtained by making $a = 0$, in which case (38) and (42) become (15) and (17) respectively. For the latter case we take h as the distance from the centre of the circle to the surface of the wall, and, substituting $a = (l-h)$ and making l infinite, we find that (38) and (42) reduce to the common form

$$S = \frac{\pi}{\tanh^{-1}\left[\dfrac{h-r}{h+r}\right]^{1/2}}. \quad . \quad . \quad . \quad . \quad (43)$$

This is as it should be, for it is well known* that the two-dimensional case of a point source near the boundary of a semi-infinite plane gives circular isothermals with a value for the shape factor as in (43).

Line Source.

For this case we use the transformation shown in fig. 4, which yields, for the relation of z and w planes, the following :—

$$\tan(z\pi/4l) = \lambda \operatorname{sn}(w/B, k). \quad . \quad . \quad . \quad (44)$$

* See, *e.g.*, Jeans, 'The Mathematical Theory of Electricity and Magnetism,' 4th ed. p. 295, ex. 73 & 78.

Along the real axis we have the isothermal passing through the points $x=(l-a-r)$ and $x=(l-a+r)$, and hence

$$\lambda \operatorname{sn}(\mathrm{U}, k)=\tan\left[\frac{\pi}{4}-\frac{(r+a)\pi}{4l}\right], \quad . \quad . \quad . \quad (45)$$

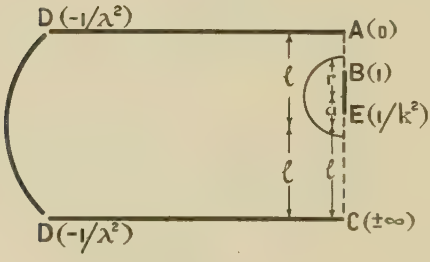
$$\frac{\lambda}{k \operatorname{sn}(\mathrm{U}, k)}=\tan\left[\frac{\pi}{4}+\frac{(r-a)\pi}{4l}\right]. \quad . \quad . \quad . \quad (46)$$

So that by multiplication and division

$$\frac{\lambda^2}{k}=\tan\left[\frac{\pi}{4}-\frac{(r+a)\pi}{4l}\right]\tan\left[\frac{\pi}{4}+\frac{(r-a)\pi}{4l}\right], \quad (47)$$

$$k \operatorname{sn}^2(\mathrm{U}, k)=\tan\left[\frac{\pi}{4}-\frac{(r+a)\pi}{4l}\right] \div \tan\left[\frac{\pi}{4}+\frac{(r-a)\pi}{4l}\right]. \quad . \quad . \quad . \quad (48)$$

Fig. 4.



Further, since the isothermal has also to pass through the end of the horizontal diameter of the circle, *i.e.*, the point $(l-a), r$,

$$\tan\left\{\frac{(l-a)+ir}{4l}\right\}=\lambda \operatorname{sn}\{(\mathrm{U}+i\mathrm{V}), k\}. \quad . \quad (49)$$

We have therefore four relations—(47), (48), and the real and imaginary parts of (49)—to determine the four unknowns k, λ, U, and V. A process of solution by trial must be adopted, and it can be shortened as follows:—

Estimate the shape factor S by taking the mean of $\mathrm{S_U}$ and $\mathrm{S_L}$ as given by (38) and (42), and find by trial the

modulus k which will give this value in the "line source" formula for S, viz :—

$$S = \frac{2K'}{\mathrm{sn}^{-1}\left[\dfrac{\tan\left(\dfrac{\pi}{4}-\dfrac{(r+a)\pi}{4l}\right)}{k\tan\left(\dfrac{\pi}{4}+\dfrac{(r-a)\pi}{4l}\right)}\right]^{1/2}} (\mathrm{mod}\, k). \quad . \quad (50)$$

Hence from (47) and (48) obtain U and λ. A value for V must now be found which will satisfy as closely as possible the two relations implied in (49). Instead, however, of finding the precise value of V for the flow line at the end of the horizontal diameter we may assume a value for V near the required value*, and calculate the coordinates of the point where this flow line cuts the isothermal by means of the following relations deduced from (44) :—

$$x = \frac{2l}{\pi}\tan^{-1}\left\{\frac{2A}{1-A^2-B^2}\right\}, \quad . \quad . \quad . \quad (51)$$

$$y = \frac{l}{\pi}\log\left\{\frac{(1+B)^2+A^2}{(1-B)^2+A^2}\right\}, \quad . \quad . \quad . \quad (52)$$

where

$$A = \frac{\lambda\,\mathrm{sn}\,(Uk)\,\mathrm{dn}\,(Vk')}{\mathrm{cn}^2\,(Vk')+k^2\,\mathrm{sn}^2\,(Uk)\,\mathrm{sn}^2\,(Vk')},$$

$$B = \frac{\lambda\,\mathrm{cn}\,(Uk)\,\mathrm{dn}\,(Uk)\,\mathrm{sn}\,(Vk')\,\mathrm{cn}\,(Vk')}{\mathrm{cn}^2\,(Vk')+k^2\,\mathrm{sn}^2\,(Uk)\,\mathrm{sn}^2\,(Vk')}.$$

Then, provided that x does not differ much from $(l-a)$ and the point of intersection lies nearly on the required circle, the solution may be regarded as satisfactory. An example treated on the lines above indicated is given in Section VI. below.

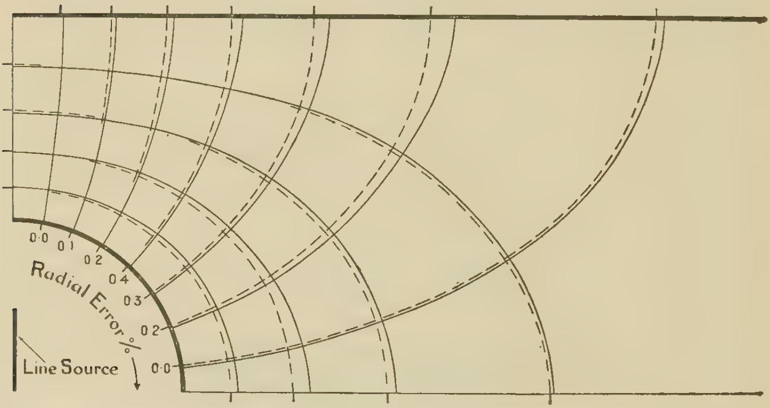
VI. Graphical Estimation of Heat Flow.

A simple method of estimating the distribution of heat flow in two-dimensional cases has been developed by Wedmore and used by him in the solution of a number of

* The value of V will be greater or less than K'/2 according as the centre of the circle (fig. 4) is above or below the midplane of the wall.

practical problems*. This method depends on the fact that flow lines and isothermals must cut at right angles, and, if drawn for infinitely small steps of temperature and heat flow, will divide the whole area to be considered into elementary rectangles having a constant ratio between the lengths of adjacent sides. The ratio may conveniently be made unity, so that the rectangles become squares. In attempting to carry out this principle on a finite scale the spaces dealt with cannot be true squares, since both isothermals and flow lines will generally be curved. However, by a process of trial and error, two sets of lines can be drawn as smooth curves cutting at right angles and complying approximately with the theoretical conditions. The correctness of

Fig. 5.



the distribution in the case of those quadrilaterals not differing greatly from squares can be judged by observing whether the sums of the lengths of the opposite pairs of sides are nearly equal ; but where the departure from the square form is considerable the quadrilaterals should be tested by subdivision to see whether they will yield approximate squares.

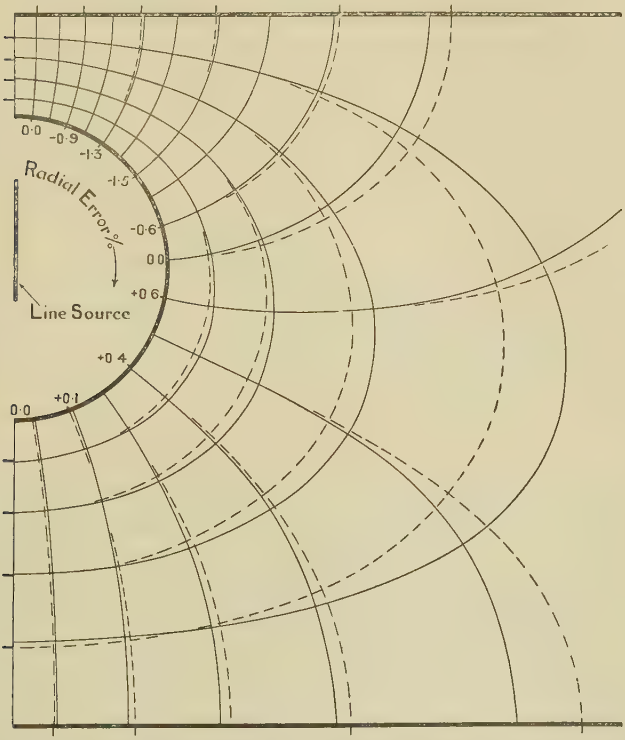
* Journ. Inst. Elect. Eng. lxi. (1923), Appendix iv., on the thermal resistance of a three-core cable; and Journ. Inst. Elect. Eng. lxviii. (1930), Appendix i., on the corner correction for an insulating layer with a right-angled bend. The correction found for the latter is in close agreement with that derived theoretically in Jeans's 'Mathematical Theory of Electricity and Magnetism,' 4th ed. p. 277.

Since this section was written and the drawings prepared a more detailed description of the method has come to hand, viz., Lehmann, *Elektech. Zeitschrift*, p. 995 (1909).

The method may involve a considerable number of trials before a satisfactory distribution is obtained, but it has the great advantage that it enables heat flow to be estimated in cases too complex for theoretical treatment, such as finite shapes of complicated outline or forms consisting of two or more materials of different conductivities.

The problem dealt with above, in which formulæ for calculating the distribution have been derived, seemed to afford a favourable opportunity for trying the method. In

Fig. 6.



order that the trial should be independent two problems were given to a worker previously inexperienced in the method, who, after some little practice, produced the drawings shown as full lines in figs. 5 and 6. The dashed lines representing the calculated distributions were added afterwards.

Fig. 5.—This is the case of a cylinder symmetrically placed in the wall and having a diameter 5/11 of the thickness of the wall. Taking a line source solution the modulus is given by equation (12), and the temperature U of the isothermal

coincident with the circle where it cuts the axes is obtained from (21). Dividing U into any convenient number of parts, *e. g.*, five, we may calculate the coordinates of the intersection of the isothermals $u = 0$, U/5, 2U/5, etc., with the flow lines $v = 0$, U/5, 2U/5, etc. This can be done by equations (19), (20), (24), and (25). The calculated distribution is shown in fig. 5 by the dashed lines, with which the full lines of the estimated distribution are seen to be in fair agreement. It should be explained that the latter was not in this case an entirely independent estimation, since some guidance as to the total heat flux was obtained from the electrical measurements described in Section VIII. below.

With regard to the departure of the inner isothermal from the true circle, this is also indicated (fig. 5). The isothermal always comes within the circle, and the amounts of the differences with reference to the radius of the circle taken as unity are as shown. It will be seen that the maximum departure in radius is of the order of 4 parts in 1000.

The generating line source is also shown on fig. 5.

Fig. 6.—In this case the cylinder is eccentric, a/l being equal to 2/7 and r/l to 3/7, where r is the radius of the circle, a the displacement of its centre from the mid-plane of the wall, and $2l$ the thickness of the wall. Then, taking a line source and calculating as explained in the preceding section, equation (44) becomes

$$\tan (z\pi/4l) = 0{\cdot}38945 \,\mathrm{sn}\,(w, 32^\circ). \quad . \quad . \quad (53)$$

The temperature of the isothermal, which is coincident with the circle where it cuts the x axis, is given by (48), and the distribution is calculated by use of (51) and (52). Again, the estimated distribution, shown by full lines, is in fair agreement with the calculated distribution, which is shown dashed, while there is an almost exact agreement between the estimated and theoretical values of the total heat flux. The latter is perhaps a coincidence, but it may be mentioned that Wedmore estimated the accuracy of the method in one of his cases to be within 2 per cent.

As before, fig. 6 shows the generating line source and indicates the percentage departure in radius of the inner isothermal as compared with the true circle.

VII. Electrical Determination of Shape Factor.

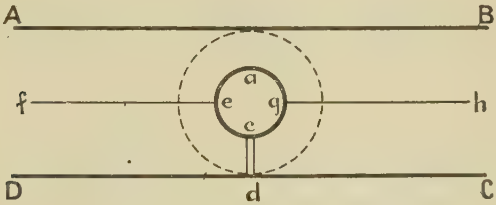
It has been explained in former papers* that the shape factor may be obtained by measurement of the electrical

* Awbery and Schofield, "Effect of Shape on Heat Loss through Insulation" (5th Inter. Congress on Refrigeration, 1928), and Schofield, Phil. Mag. x. p. 495 (1930).

resistance of a sheet of metal cut to the appropriate shape, and it has been pointed out that, since the systems of isothermals and flow lines in any case are interchangeable, the electrodes used for measurement can be made to correspond either with the boundary isothermals or flow lines according to which is most convenient. Let us consider the application of this method to the problem treated here, and take as an example the case of a cylinder symmetrically placed in a wall (fig. 7).

Then if ABCD is a long sheet of metal, one electrode could be attached along AB & CD which would be at the same potential and the other round the circumference of the circle *aecg*. Alternatively the figure could be cut in half longitudinally and the electrodes attached along AB and the corresponding semicircle. The resistance in either case would be inversely proportional to the required shape factor.

Fig. 7.



It is clear that with either of the above-mentioned arrangements the accurate attachment of the electrodes would be a matter of very considerable difficulty.

However, by inverting the problem we may make the electrodes correspond with any pair of flow lines provided they are straight, and in the present case this can conveniently be done by cutting out the circle *aecg* from the sheet and cutting through the line *cd* *. If now electrodes are attached along both sides of *cd*, the required conditions will be closely fulfilled. In making the measurements the procedure previously described was closely followed. Thus brass sheet 1·6 mm. in thickness was used, and the electrodes were thick brass strips soldered along the lines indicated. The actual resistance measurement was made by a potential method between the lines *ef* and *gh*. These two lines were first explored by two needle points connected to a potentiometer,

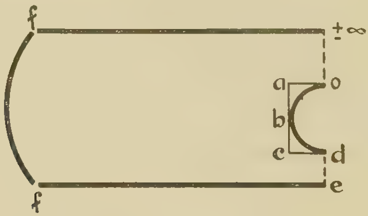
* This plan was suggested by Mr. D. E. A. Jones, who made the electrical measurements.

to prove that they were strictly equipotential in accordance with the requirements of the problem. Having made the measurement, it remains to compare the resistance with that of a rectangle cut from the same sheet or some other appropriate shape of which the shape factor is known. In the present case by cutting down the sheet round the dotted circular line a direct comparison can be obtained with the well-known case of concentric circles. It may be added that the shape factor given by the resistance measurement for the case illustrated in fig. 5 was found to be 6·07 as against 6·12 * obtained by calculation from a line source.

VIII. An Alternative Method of Theoretical Treatment.

Various suggestions have been made for the treatment of curved boundaries by the method of conformal representation,

Fig. 8.



and reference may be made here particularly to the work of Page † and Richmond ‡. We may apply the method of the latter to the case of a cylinder eccentrically placed in an infinite wall (see fig. 8). Then, if the values of t at the angular points are as shown, the transformation

$$\frac{dz}{dt}=\frac{\{(t-b)^{1/2}+\mathrm{A}t^{1/2}\}\{(t-b)^{1/2}+\mathrm{B}(t-d)^{1/2}\}}{(t-f)(t-e)^{1/2}(t-d)^{1/2}t^{1/2}} . \quad (54)$$

is obtained from the figure, o, a, c, d, e, f, etc. by replacing the terms $(t-a)^{1/2}$ and $(t-c)^{1/2}$ which would normally occur in the numerator by the two terms shown. The effect is to make dz/dt complex for values of t between o and b and also between

* This value applies to an infinitely long strip. The length of the brass strip was 2·6 times its width, and this can be shown from equation (8) to be practically equivalent to an infinitely long strip.

† Proc. Lond. Math. Soc. ii. p. 313 (1912).

‡ *Ibid.* xxii. p. 389 (1923).

b and d, so that a curved boundary obd is obtained in place of the rectangular boundary $oacd$. The curvature is controlled by the constants A and B, and hence the transformation allows an approximation to the required circle to be obtained. The process of solution involves the integration of (54), which apparently yields an expression of six terms, including three elliptic integrals of the third type. This is of course very much more complicated than the single $\tan^{-1}$ function * which is obtained from a line source and which gives a satisfactory approximation. Similarly for a centrally placed cylinder the line source gives the simpler solution. It should be added that with either method the transformation from the w to the t plane is of the same form.

Though not specially convenient in the case now under consideration, the method outlined above is obviously one of wide application †.

In conclusion, the author wishes to acknowledge his indebtedness to Mr. D. E. A. Jones, Observer at the National Physical Laboratory, for the electrical measurements and for the drawing of figs. 5 and 6.

XXVIII. *A new Electron-Inertia Effect and the Determination of* m/e *for the Free Electrons in Copper.* *By* S. J. Barnett, *Professor of Physics in the University of California at Los Angeles, and Research Associate of the California Institute of Technology* ‡.

[Plates VI. & VII.]

§ 1. In Maxwell's 'Treatise on Electricity and Magnetism' (§§ 574, 575, and 577) he describes three inertia effects which should exist in conductors if the electric current is due to the motion of one kind of electricity only and if this electricity has inertia. Special cases, as follows, will suffice to illustrate these effects, on all of which Maxwell appears to have tried experiments :—

(1) If the current in a circular or cylindrical coil of wire free to move about its axis is altered, the free electricity will be accelerated and the coil itself will be accelerated in

* See left-hand side of equation (44).

† See, *e. g.*, Cockcroft, Journ. Inst. Elect. Eng. lxvi. p. 403 (1928).

‡ Communicated by the Author. Revision of a paper presented to the American Physical Society at the Eugene Meeting, June 18, 1930.

the opposite direction, the changes of angular momenta being equal in magnitude and opposite in sign. This is the effect investigated in this paper, and now observed for the first time.

(2) If the coil of wire is traversed by a steady electric current the electricity has a constant angular momentum about the axis, and the coil should exhibit the properties of a gyrostat. Maxwell looked for this effect in or about 1861, but without success, on account of the great experimental difficulties. The experiments on magnetization by rotation which were first made and presented to the American Physical Society by me in 1914, however, established the same effect for the individual whirls of Ampère, each of which behaves as Maxwell's coil would behave if suitable conditions could be obtained *.

(3) If the coil of wire is accelerated about its axis the free electricity will be differently accelerated, lagging behind when the coil's speed is increased, and going ahead when the speed is decreased. Thus the acceleration of the coil gives rise to an electric current in it. This is the effect discovered by Tolman and Stewart in 1916 and investigated in three papers by Tolman and Stewart †, Tolman, Karrer, and Guernsey ‡, and Tolman and Mott-Smith §.

§ 2. While all these effects have greatly interested me for many years, the investigation described here on effect (1) would not have been made at the present time except for the fact that this effect enters as a small correction in an elaborate investigation of the Einstein and de Haas effect (rotation by magnetization) which I have had under way for some time. It was important to be entirely certain about this correction, which could be studied with but little addition to the equipment already long in satisfactory operation for the other work. This paper is thus a by-product of the other investigation, an account of which it is hoped to publish soon ‖. To this the reader must be referred for many details.

§ 3. Two methods of experimentation have been used, a null-graphical method and a deflexion method, both closely related. The former will be described first. A diagram

* Phys. Rev. vi. p. 239 (1915).
† Phys. Rev. viii. p. 97 (1916).
‡ Phys. Rev. xxi. p. 525 (1923).
§ Phys. Rev. xxviii. p. 794 (1926).
‖ *Note added in proof.*—See Proc. American Academy of Arts and Sciences, lx. no. 8, pp. 273–347 (1931).

illustrating the methods of experimentation is given in fig. 1.

The coil has the form of a long solenoid, wound on a cylinder of brass or glass. The coil and the cylinder on which it is wound will together be referred to as the rotor. The torque on the rotor produced by changing the current has its effect in producing rotation annulled by another torque, as suggested by Chattock in another connexion. This annulling torque is due to the action of the electric current traversing a coil of wire on a small permanent magnet attached to the rotor, an effect precisely calculable.

Fig. 1.

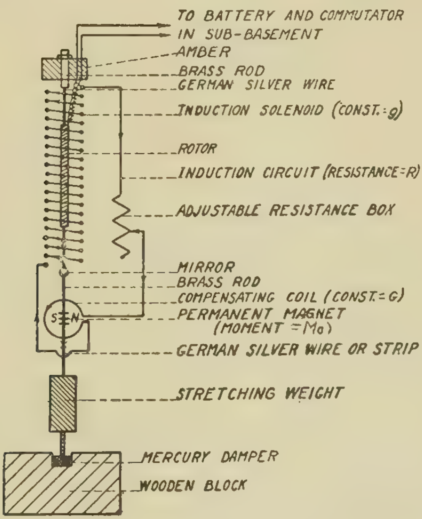


Diagram illustrating methods of experimentation.

The rotor is axially suspended by a thin German-silver wire or strip attached above to a brass rod passing through a torsion head of amber. Hanging from the lower end of the rotor by an essentially rigid joint is a small brass rod carrying on opposite sides a pair of parallel mirrors, and below, a group of small permanent magnets, all turned in the same direction. Hanging from the lower end of the brass rod, by another German-silver suspension, is a stretching weight, which keeps the axes of the rods in line and the suspensions tight. A small rod projecting downward from

the weight dips into some mercury in a wooden cup, and helps to damp out any lateral motion of the hanging bodies.

Current is led to and from the coil by two fine wires twisted together and soldered to small brass rods passing through the torsion head and connected to a storage battery through a rotating commutator, or to an alternating current sine-wave generator.

When the commutator or generator operates, the rotor coil is acted on by an alternating torque and is set into vibration. In order to make the vibration as great as possible, the speed of the commutator is adjusted and controlled by elaborate devices in such a way that resonance occurs and is steadily maintained. The range of the vibration is examined by lamp, mirror, lens, and scale, the scale being more than $4\frac{1}{2}$ metres distant from the mirror.

The rotor is surrounded with a much longer and uniformly wound coil of wire, which will be called the *induction solenoid*, or by a small coil with bunched windings, the *induction coil*, at its centre; and the group of small permanent magnets is at the centre of a small double coil, the *compensating coil*, the planes of whose wires are parallel to the magnets. Currents sent in opposite directions through this coil will produce opposite torques upon the magnets and thus upon the vibrating system. So that, if the currents are properly timed, and of the proper magnitude, these torques will just annul the electron inertia effect.

This condition is brought about as follows:—The induction solenoid or coil is connected through an adjustable resistance box in circuit with the compensating coil. When the current in the rotor changes, it induces at the same time a current in this induction circuit, whose magnitude can be increased or decreased at will by changing the resistance R (or the conductance X) of the box. Thus, at the same time as an electron-inertia torque acts upon the rotor system, a torque of electromagnetic origin, which may be made exactly equal to it, is given to the system. If the terminals of the compensating coil are connected in one way to the rest of the circuit, the two torques will be in the same direction and the resultant torque twice one of them; if the connexions are reversed the two torques will annul one another, and the vibration will disappear. From the directions in which the coils are wound, and the permanent magnet turned, it is easy to predict in advance how the terminals should be connected to produce the compensation, provided we know the sign of the free electricity. The connexions actually found necessary always show that the electricity is negative.

If R_0 is the resistance and $X_0=1/R_0$ the conductance of the circuit when the vibration is annulled, m_0 the moment of the group of permanent magnets, and G and g the constants of the compensating coil and induction solenoid, it is easy to show that

$$-2\frac{m}{e}=\frac{Ggm_0}{R_0}=Ggm_0X_0, \quad . \quad . \quad . \quad (3,1)$$

provided that Maxwell's assumptions are correct and m and e are the mass and charge, respectively, of a particle of the mobile electricity.

§ 4. Formula (3, 1) is readily obtained as follows: Let μ denote the magnetic moment of the rotor, j' the angular momentum of the free electricity, and j the angular momentum of the rotor; and let ρ denote the ratio j'/μ, so that $j'=\rho\mu$.

The torque upon the rotor due to the changing current is then

$$t=\frac{dj}{dt}=-\frac{dj'}{dt}=-\rho\frac{d\mu}{dt},$$

provided that the rotor is maintained at rest, as in the null method at the critical time.

The magnetic flux ϕ through the induction solenoid is $\phi=g\mu$. Thus when μ changes there is induced in the induction circuit the electromotive force $-\frac{d\phi}{dt}=-g\frac{d\mu}{dt}$, which produces the current $c=-g\frac{d\mu}{dt}\Big/R$, provided that the reactance of the circuit is negligible, as it is in fact. The torque upon the rotor system due to this current is

$$Gm_0c=-Ggm_0\frac{d\mu}{dt}\Big/R.$$

When the two torques are equal in magnitude and opposite in sign $R=R_0$ and $\frac{dj}{dt}=-Gm_0c$. This gives

$$-\rho=\frac{Ggm_0}{R_0}.$$

For an isolated electron moving uniformly in a circular orbit the ratio of angular momentum to magnetic moment is $\frac{2m}{e}$. If in the conductor the free electricity consists of

electrons, and if the same ratio of mass to charge exists, we shall have

$$-\rho = -\frac{2m}{e} = \frac{Ggm_0}{R_0} = Ggm_0X_0, \quad . \quad . \quad (4, 1)$$

which is the result already quoted (3, 1).

There may, of course, be disturbing torques in quadrature or in phase with (or in opposition to) t. In the first case the amplitude can be reduced to a minimum, but not to zero, by changing X; in the second it will come to zero, but the value X_0 inserted in equation (4, 1) will give not ρ but $\rho + \Delta\rho$, where $\Delta\rho$ is the error.

If the amplitude A is observed as a function of R for a number of resistances on both sides of R_0, and reckoned negative on one side and positive on the other (since the torque changes sign at this value of R), and if A is plotted as a function of $X = 1/R$, it is easy to show that the result will be a straight line, provided there are no disturbing torques in quadrature with t and that the effect of amplitude, discussed in § 5, is negligible. If there is a torque in phase with t we shall have

$$-\rho + \Delta\rho = Ggm_0X_0,$$

where $\Delta\rho$ is the resulting error in ρ, as above. If the experiments can be repeated with the phase of the disturbing torque reversed, we shall have

$$-\rho - \Delta\rho = Ggm_0X_0',$$

where X_0' is the new value of the conductance for zero amplitude. Then

$$\rho = Ggm_0\left(\frac{X_0 + X_0'}{2}\right). \quad . \quad . \quad . \quad . \quad (4, 2)$$

If there are torques in quadrature with t, a minimum amplitude not zero will result, as stated above, and the line will curve on each side near the minimum, where it becomes discontinuous.

If the induction coil with bunched winding is used, in place of the induction solenoid, it is easy to show that the constant g must be replaced by the constant $g' = \frac{4\pi Z_2}{l}$, where Z_2 is the total number of turns in the bunched winding and l is the axial length of the uniform winding on the rotor.

§ 5. To obtain a more general theory we may proceed as follows: Suppose the rotor coil to contain Z turns of thin

wire with cross-section a and mean radius r, and to be traversed by a current

$$i = \mathrm{I} \sin \omega t. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5, 1)$$

Let the number of free electrons per unit volume be denoted by n, and the angular amplitude of the rotor's vibration by Θ, its angular velocity $\dot{\theta}$ (in phase with the angular momentum j) being given by the equation

$$\theta = \omega\Theta \sin(\omega t + \beta). \quad . \quad . \quad . \quad . \quad . \quad (5, 2)$$

If v denotes the electron velocity and V the velocity of the wire, the current in the coil will be

$$i = nea(v - \mathrm{V}),$$

so that
$$v = \frac{i}{nea} + \mathrm{V} = \frac{i}{nea} + r\dot{\theta} \quad . \quad . \quad . \quad . \quad . \quad (5, 3)$$

and
$$\dot{v} = \frac{\dot{i}}{nea} + r\ddot{\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5, 4)$$

Thus the torque on one electron, reckoned about the axis of symmetry of the coil, will be

$$rm\dot{v} = \frac{rm}{nea}\dot{i} + r^2 m\ddot{\theta}, \quad . \quad . \quad . \quad . \quad (5, 5)$$

and the torque on all the electrons in the Z turns will be equal to this quantity $\times 2\pi ranZ$. Hence the total inertia torque on the rotor, equal in magnitude and opposite in sign to that on the electrons, will be

$$t = -\frac{2m}{e} . \pi r^2 \mathrm{Z} . \dot{i} - \frac{2m}{e} \pi r^2 \mathrm{Z} . near\ \ddot{\theta}. \quad . \quad (5, 6)$$

In conformity with (5, 1) and (5, 2) this equation may be written

$$t = -\frac{2m}{e} . \pi r^2 \mathrm{Z} . \mathrm{I}\omega \cos \omega t - \frac{2m}{e} \pi r^2 \mathrm{Z}\ near\ \omega^2 \Theta \cos(\omega t + \beta)$$

$$= \mathrm{T}_1 \cos \omega t - \mathrm{T}_2 \cos(\omega t + \beta). \quad . \quad (5, 7)$$

In the null method T_2 vanishes, and (5, 7) gives the result already obtained.

In these experiments the ratio $\mathrm{T}_2/\mathrm{T}_1 = \dfrac{near\ \omega\Theta}{\mathrm{I}}$ was very small for the amplitude produced by the inertia effect, viz., about $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{20} \div 460$ radian. Thus if we take $n = 10^{23}$,

$e=1\cdot6\times10^{-20}$, $a=5\times10^{-5}$, $r=0\cdot27$, $\omega=2\pi\times14\cdot6$, $I=0\cdot01$, all in C.G.S. and E.M. units, and Θ as above, we obtain for the ratio about 5×10^{-3}. For the largest amplitudes used in the experiments (in the calibrating part of the deflexion method), viz., amplitudes of the order of 150 times that produced by the inertia effect, the ratio is of the order of unity.

When Θ or T_2 does not vanish it is easy to show that the relation

$$\beta=\cot^{-1}\frac{T_2}{T_1}. \quad . \quad . \quad . \quad . \quad . \quad (5,8)$$

holds approximately, so that t_2 is a quarter cycle ahead of t_1.

Since T_2/T_1 increases with the frequency and amplitude, it is perhaps possible that by making experiments in which the amplitude is made large (*e.g.*, by an independent current in the compensating coils, or, in ths case of magnetic substances, by the gyromagnetic torque) T_2 may be measured, and thus information, which is badly needed, on the quantity n be obtained. The probability of success in an experiment of this kind is, of course, greatly diminished by the fact that t_2 is in quadrature with t_1.

§ 6. When the amplitude is small T_2 is negligible in comparison with T_1, and the null method may be replaced by an amplitude or deflexion method, as follows:—

Let the first harmonic of the current traversing the rotor coil be $i=I\sin\omega t$. Then the amplitude of the first harmonic of the inertia torque on the rotor, which is the only harmonic that is effective on account of resonance, is

$$T_e=\rho\pi r^2 Z\omega I=|\rho|\omega M,$$

where M is the amplitude of the coil's moment. If the induction circuit is open this torque will produce a vibration with the amplitude A_e.

With the rotor coil and the induction solenoid both open, let a current whose first harmonic is $c=C\sin(\omega t+\alpha)$ be sent through the compensating coil. This will produce a torque whose amplitude is $T_c=Gm_0C$; and this torque will produce a vibration amplitude A_c. We have then

$$\frac{T_e}{T_c}=\frac{|\rho|\pi r^2 Z\omega I}{Gm_0C}=\frac{A_e}{A_c},$$

and

$$|\rho|=\frac{A_e}{A_c}\cdot\frac{Gm_0C}{\pi r^2 Z\omega I}=\frac{A_e}{A_c}\cdot\frac{Gm_0C}{\omega M}. \quad . \quad . \quad . \quad (6,1)$$

§ 7. There are also electron-inertia torques upon the rotor, if conducting, due to the induction of currents therein by the changing magnetic flux; but it is easy to show that they are very much less than those just considered. One of the two terms vanishes in the null method; both are in quadrature with t in the deflexion method.

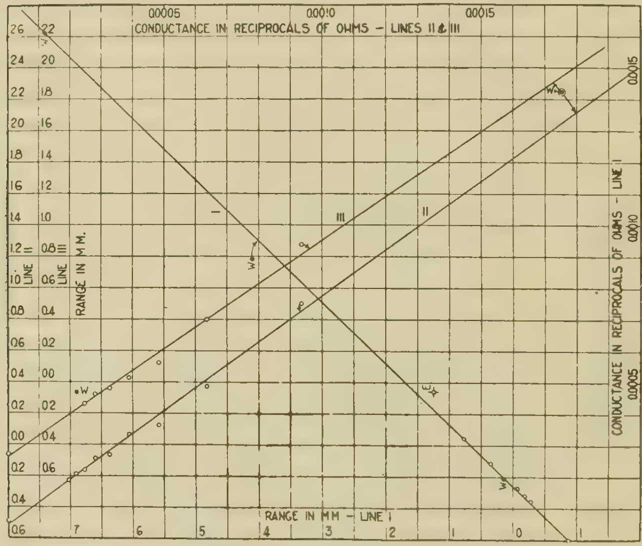
§ 8. In order to avoid serious disturbances due to the steady part of the earth's magnetic field the rotor was mounted at the centre of three suitable electric coils traversed by such currents as to neutralize the three components of the earth's intensity: and to avoid the effects of vibration and of the fluctuations in the earth's field all of the vibration experiments were made after one o'clock at night. The effects of vibration of the building were greatly reduced by hanging the chief part of the apparatus from the ceiling by four sets of springs. The arrangement is shown in Pl. VI.

§ 9. A number of sets of observations, by both methods, have been made on two rotors, one of brass, the other of glass. The first was 0·24 cm. in diameter, and wound with two layers of SSC copper wire B. & S. No. 40, in a coil about 27 cm. long. The second was 0·48 cm. in diameter; it was wound with four layers of the same kind of wire. The coil in this second rotor was 23·2 cm. long. A photograph of the complete rotor, in two parts, is reproduced in Pl. VII.

While both rotors have given results of the sign and order of magnitude to be expected, by far the most reliable observations are some of those with the glass rotor. On three nights, especially, the disturbances were unusually small, and excellent results were obtained. In each set of observations the amplitude was determined on a regular time schedule for each of a number of suitable resistances of the induction circuit, both increasing and decreasing, on both sides that which would give zero or minimum amplitude. After the amplitude was determined for any value of the resistance, the connexions between the rotor coil and the generator were reversed, and the amplitude re-determined. Then the whole series was repeated with the torsion head (and vibrator magnet) turned in azimuth through 180°, and the connexions between the compensating coil and the rest of the induction circuit reversed. In the experiments on magnetic rotors with coils wound upon them the first reversal was made chiefly to eliminate the effects of magneto-striction; the change in azimuth and second reversal, to eliminate the effects of the residual parts of the earth's field. It is not clear just how

either of these changes can affect the value of ρ determined from the observations now under consideration; and the first at least ordinarily does not in the case of the glass rotor, while it did in the case of brass. The second may or may not produce a decided effect. It produced no appreciable effect in the experiments of May 6 ($X_0 = X_0'$). In those of Sept. 8 and 10 the effect of the change was greater than 20 per cent. The concordance of the results obtained on these latter days, by taking the means of X_0 and X_0' on each occasion, with those of May 6 justifies the procedure.

Fig. 2



Relation between vibration range and conductance.

The curves for the three days mentioned are given in the chart (fig. 2)*. The values of the conductances are given directly by the chart. The observations of May 6 were made with the small induction coil with bunched winding, whose constant $g' = 4\pi Z_2/l = 4\pi \times 879 \div 23{\cdot}2$ e.m.u. The values of the other constants are as follows:—$G = 1832$, $m_0 = 0{\cdot}773$, both in e.m.u. Line I. of the chart is drawn for these observations. They were made with rectangular waves.

* Points marked W or E were obtained for one azimuth of the vibrator magnet only—N.P.E. or N.P.W.

After these observations were made this work had to be set aside for several months. When it was resumed a disturbance occurred which it seemed possible was in part due to looseness of the winding. The rotor was therefore rewound as nearly as possible in the same way, except that the wire was varnished during the winding. The disturbance referred to, which apparently contained a torque steadily in opposition to the electron-inertia torque, persisted and was not eliminated until the frequency was increased by replacing he lower suspension with a stiffer one.

Line II. of the chart gives the observations of Sept. 10; Line III. those of Sept. 8. They were made with sine waves, and with the induction solenoid, whose constant is 217 e.m.u., replacing the induction coil with bunched winding. The other constants needed for the calculations of ρ remained unchanged.

From the data now given the following values of ρ (all in e.m.u.) are obtained:—May 6, $1{\cdot}15 \times 10^{-7}$; Sept. 8, $1{\cdot}06 \times 10^{-7}$; Sept. 10, $1{\cdot}09 \times 10^{-7}$; mean $1{\cdot}10 \pm 0{\cdot}03$.

The observations of May 6 are probably the most reliable, as all disturbances were almost completely absent while they were made. It is possibly significant that the result for this date is still closer than the others to the standard value of $2m/e$, viz., $1{\cdot}13 \times 10^{-7}$ e.m.u.

§ 10. The deflexion method of § 6 has also been used in connexion with the graphical-null method; but the results are much less reliable, partly on account of the larger number of quantities involved, partly * on account of the changes occurring between the different parts of the observations, and perhaps also partly for other reasons which are obscure.

* In making a set of experiments a current of 100 milliamperes (r.m.s.) with the frequency of 50 ∼/sec. was sent through the rotor coil for a long time before the observation of vibrations began. Then, with this current unchanged, the rotor was tuned by sending the current C with frequency ν through the compensating coil and adjusting the tuning-fork control until the maximum amplitude A_c was obtained. Then from the same source, with frequency ν, a current of 100 milliamperes (r.m.s.) was sent through the rotor coil and the vibrations observed for one half set. Then the 100 milliampere 50 ∼/sec. current was again sent through the rotor coil and C and A_c again determined without retuning. The 100 milliampere currents heated the rotor considerably; the change from one current to the other required a little time; and it is extremely improbable that the different currents were ever so nearly equal as to eliminate error from this source; it is therefore not surprising that the results fail to agree exactly with those obtained by the null method.

The constants G and m_0 have already been given. The amplitudes A_e are taken from the chart. These latter and the remaining quantities involved in equation (6, 1), including the resulting values of ρ, are given in the table.

§ 11. Both methods of experimentation thus give for the ratio of the mass of the electron to its charge in the copper conductor values which are, within the probable experimental errors, equal to the ratio determined for free electrons. The agreement is especially close with the null method, which is much the more reliable. It is probable that, with apparatus designed throughout especially for work on the problem treated here, more precise results could be obtained. Inasmuch, however, as it will be impracticable for me in the near future to undertake an investigation requiring such apparatus, the approximate but conclusive results already obtained are given here.

Date 1930.	$\nu=\frac{\omega}{2\pi}$.	M †.	C.	A.	A_e.	ρ.
	~ /sec.	e.m.u.	e.m.u.	cm.	cm.	e.m.u.
May 6 ...	6·73	$\frac{4}{\pi}\times 14{\cdot}35$	$\frac{4}{\pi}\times 3{\cdot}15\times 10^{-6}$	7·0	0·086	$0{\cdot}90\times 10^{-7}$
Sept. 8 ...	14·6	$\sqrt{2}\times 14{\cdot}35$	$\sqrt{2}\times 1{\cdot}04\times 10^{-5}$	5·86	0·048	$0{\cdot}91\times 10^{-7}$
Sept. 10 ...	,,	,,	$\sqrt{2}\times 1{\cdot}03_5\times 10^{-5}$	7·17	$0{\cdot}051_6$	$0{\cdot}80\times 10^{-7}$

† While M can be roughly calculated from the dimensions of the rotor and coil and the number of turns in the latter, a direct determination by experimental comparisons with magnetic standards is greatly to be preferred. The value of M given in the table was determined in this way after the rewinding. The difference in its values before and after rewinding cannot be great.

§ 12. I am under obligations to many individuals and organizations without whose help this work, and that of which it is a by-product, would have been impossible. Together with the institutions of learning mentioned below, I am especially indebted to the Carnegie Institution of Washington, which has furnished most of the fixed equipment; to Mr. Wm. A. Arnold of the Institute, who, as research assistant, has given me indispensable help of many kinds; and to Mr. G. H. Jung, of the University, who has done most of the finer mechanical work. The chief experimental work has been done in the Norman Bridge Laboratory of the Institute.

The University of California at Los Angeles and
The California Institute of Technology.
November 18, 1930.

XXIX. *The Influence of Hydrostatic Pressure on the Critical Temperature of Magnetization for Iron and other Materials.* *By* L. H. Adams and J. W. Green *.

THIS investigation was undertaken principally because of its bearing on the earth's magnetic field and its relation to the problem of the central iron core. It is now generally accepted that within the earth is a metallic core which consists mainly of iron with a few per cent. of nickel, and has a diameter of about 6000 kilometres, or a little less than one-half the diameter of the whole earth. So large an amount of iron, if it were in the usual ferromagnetic state, would have a profound influence on the magnetic field observed at the surface, and would be one of the major factors entering into any theory of the earth's magnetic field. When it is remembered, however, that iron loses its magnetic quality at a temperature less than 800°, and that in the interior of the earth the temperature, while difficult to estimate satisfactorily, is certainly of the order of some thousands of degrees, it would seem at first thought that iron in the interior would have no effect on the magnetism of the earth. On the other hand, the high pressure within the earth might possibly offset the effect of high temperature. The pressure at various depths below the surface may be estimated quite satisfactorily; at the top of the iron core it is 1,800,000 atmospheres, and at the centre 3,200,000 atm. These pressures are of such magnitudes that it is evident that even a small specific effect of pressure on the critical temperature of magnetization might suffice to maintain the iron core in the ferromagnetic state. It becomes of importance, therefore, to make any possible measurements on the change of the magnetic inversion-point with pressure.

In beginning this investigation the authors were not unmindful of the difficulties which would arise—difficulties not only of experimentation, but, more especially, of interpretation and application. The highest pressure ever attained for purposes of exact measurements is 20,000 atm., and that only at comparatively low temperatures. At high temperatures experimental technique has

* Communicated by A. L. Day, Ph.D.

progressed only so far as to permit investigations up to about 4000 atm. This is a very small part of the maximum pressure in the earth, so small a part indeed that extrapolation from the experimental range of pressures to those pressures existing at or near the centre of the earth would be without quantitative significance. It was noted, however, that of the three possibilities concerning the pressure coefficient of the inversion-temperature, viz., that it should be positive, zero, or negative, only the first would fail to lead to definite conclusions concerning the interior, since if the inversion-point were lowered by pressure or unaffected by it, there would be good reason for believing that the interior temperature is above the critical temperature of magnetization, and that the iron core is non-magnetic.

Thus the proposed investigation, which was of interest from the purely physical standpoint, seemed likely also to yield valuable results to geophysics. Accordingly, an extensive series of measurements was carried out with five ferromagnetic materials, iron, nickel, magnetite, nickel steel, and meteoric iron.

Description of Materials used.

The iron was from three different sources. Very pure electrolytic iron was obtained from the U.S. Bureau of Standards, and Swedish iron and Armco iron from commercial sources. Since no significant difference was detected in the behaviour of these three samples under pressure, the results have been lumped together under "iron."

The **nickel** was from a piece of commercial rolled sheet. It was not analysed, but was believed to be very pure.

The **magnetite** was from a large crystal of natural magnetite which was furnished by the U.S. National Museum.

The **nickel steel** was from a small bar furnished by the U.S. Bureau of Standards. It contained 35 per cent. nickel, a small amount of carbon, and a trace of chromium.

The **meteoric iron** was taken from a piece of the Canyon Diablo meteoric iron, which was furnished us by the U.S. National Museum. The composition, according to J. E. Whitfield *, is as follows :—Fe (metallic), 89·17 per

* *Vide* Merrill, Am. J. Sci. xxxv. p. 513 (1913).

cent.; $FeCl_2$, 0·10 per cent.; iron oxides, 2·52 per cent.; Ni, 7·34 per cent.; Co, 0·51 per cent.; Cu, 0·02 per cent.; P, 0·26 per cent.; S, 0·01 per cent.; C (combined), 0·11 per cent.; C (graphitic), 0·03 per cent.; Si, trace. There are also present minute amounts of other elements.

Apparatus and Method.

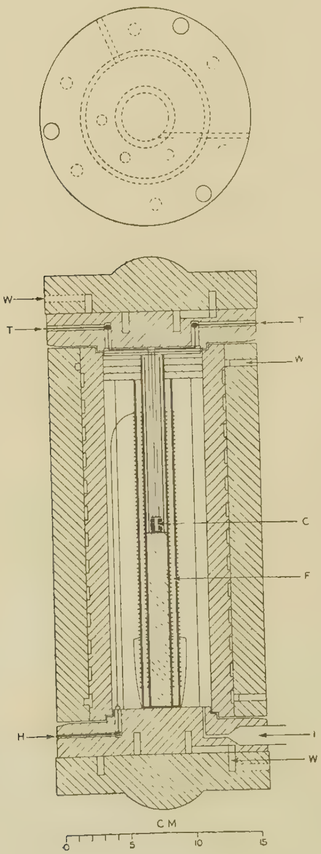
The specimens were subjected to the combined action of high pressure and high temperature in an apparatus which had been used at the Geophysical Laboratory in previous investigations *. The electrically-heated pressure-container or bomb is shown diagrammatically in fig. 1. Essentially it consists of a cylinder of vanadium steel made in two parts, with the outer cylinder shrunk on the inner one, which has an internal diameter of 75 mm. A spiral groove between the two parts allows the circulation of cooling-water. The bomb is closed on both ends by lids which also are made in two parts for ease in constructing the water-cooling passages. Through the upper lid pass five electrically-insulated leads, three of which are for the thermocouples used to measure the temperature within the bomb, the other two providing the means for an additional electrical circuit; and through the lower lid pass the leads (of phosphor-bronze wire 3 mm. in diameter) for the electric furnace, which was made by winding platinum wire, 0·6 mm. in diameter, on a thin-walled "alundum" tube of 17 mm. bore, covering the wire with a paste made of alundum cement and water, and then winding another layer of platinum wire which also was covered with alundum cement. The space between the furnace and the inner wall of the bomb was filled with granular alundum.

In order to insulate the furnace and thermocouple leads which pass into the bomb, the hole for each lead is enlarged on the inside wall to a diameter of 6 mm. for the thermocouple-leads and 8 mm. for the furnace-leads to a depth 3 times the enlarged diameter. The wire is surrounded by a close-fitting annular shell of insulating material which is then rammed firmly into place by means of an appropriate tool and a press capable of exerting a thrust of about 2000 kg. The insulating material consists,

* Smyth and Adams, J. Am. Chem. Soc. xlv. pp. 1167–84 (1923).

first (at the bottom of the hole), of a layer of fine-grained limestone (lithographer's stone), next a thin sheet of rubber, and, finally, a layer of talc or a natural talc-tremolite mixture. Inside the furnace the thermocouple wires are insulated by slipping over them short lengths of fine tubing of silica-glass, which happens to be the only

Fig. 1.



Cross-section and plan of electrically-heated pressure-apparatus. The ends are held on by means of a hydraulic press. Current is supplied to the platinum furnace-winding F through insulated connexions H. The walls, and also the ends, are cooled by water that is forced into channels at W. Carbon dioxide is pumped into the bomb at P. The induction-unit is shown at C, connected to three of the five wires which pass through insulated connexions T at the top.

practicable refractory material that is a good electrical insulator at high temperatures.

The lids are held on by means of an ordinary hydraulic press which is adjusted to produce a thrust of 30 to 50 tons greater than the thrust exerted on the lids from within the bomb, the joints being made pressure-tight by thin copper gaskets. This method of closure has a great advantage over the usual method of bolting on the lids, since for a given pressure within the bomb the closing force may be readily adjusted to the amount required.

The medium for transmitting pressure was carbon dioxide, which was the ordinary commercial grade supplied as liquid in steel containers. An analysis * showed it to be very pure ; the residue, after absorption of the CO_2, was about 0·1 per cent., and consisted of 95 per cent. nitrogen, 2 per cent. methane, and 3 per cent. hydrogen. The carbon dioxide was pumped into the bomb through a pipe connected to the lower lid. This was accomplished by means of an oil-pump which was connected to the low-pressure side of a "pressure intensifier" of the usual type, the high-pressure side being connected to the carbon dioxide tank and to the bomb.

Temperature in the bomb was measured with platinum-platinrhodium thermocouples and a potentiometer, the sensitivity being such that one division on the scale was equivalent to 0·1°. Pressure was measured on a Bourdon gauge to within 20 bars (metric atmospheres), which was ample sensitivity in view of the expected magnitude of the effect under investigation.

In order to maintain a high temperature in the bomb, a large amount of electrical power is required (about 2 kilowatts for a temperature of 800°). Although carbon dioxide at all temperatures above 32° is a gas, at high pressures the density is comparable with that of a liquid, and many other of its properties resemble those of a liquid rather than a gas. In the bomb the principal source of heat-loss seems to be through convection of the fluid which transmits the pressure. The task of obtaining a high temperature is here much like that which would be encountered if a heater placed in a vessel of water had to be raised to a temperature of 1000°. Especial care was taken to reduce convection as much as possible and yet

* For this the authors are indebted to Dr. E. T. Allen.

not allow too much opportunity for conduction. Not only were all the empty spaces between the furnace-winding and the wall of the bomb filled with a granular refractory material—60-mesh alundum was found satisfactory—but also the furnace cavity was filled at the top and bottom with alundum disks, and the specimen itself was imbedded in crushed silica-glass after it had been placed in a silica-glass tube which fitted tightly into the furnace. It was also found that much more satisfactory performance was obtained by the somewhat paradoxical process of cooling the bomb with hot water. Unless the circulating water was at a temperature considerably above room-temperature (preferably at 80° to 100°) the furnace temperature as read by the thermocouple was likely to change in a very erratic manner. The reason for this is not quite clear, but seems to be related to the great density changes produced when the carbon dioxide comes in contact with the cold inner wall of the bomb, and to the possible dropping of condensed masses of fluid into the furnace and on the thermocouple junction.

Unless all of these precautions are observed, annoying fluctuations of temperature, amounting to as much as 20° in a few seconds, may take place. At best the thermal conditions within the bomb are not nearly so satisfactory as in an electric furnace in air at atmospheric pressure. The temperature along the axis of the furnace varies considerably, rising to a maximum about half way between the top and the bottom; but, by locating the place of maximum temperature, or "hot spot," and placing the specimen approximately at this position, it is possible to reduce the temperature difference over a distance of 1 cm. to less than 5°. If the phenomenon under investigation can be confined to the vicinity of the thermocouple-junction, it may be expected that an accuracy of 1° may be obtained, but not much better than that. The situation may be improved by making a very large number of observations on each material, which will tend to reduce somewhat the accidental error of the final effect, and this procedure was followed in the present instance.

The method for determining when a specimen exposed to pressure within the bomb had become non-magnetic consisted in measuring the inductive effect in the secondary of a miniature transformer the core of which was made of the material under investigation. The construction is

shown in fig. 2. Round bars 10 mm. long and 2 mm. in diameter formed the sides. They were turned with a shoulder on each end, and were fitted snugly into holes drilled in the end-pieces, which were flat plates 10 mm. long, 4 mm. wide, and 1 mm. thick.

In the case of magnetite some difficulty was experienced in making the round bars. Satisfactory results, however, were obtained by using a transformer the side-rods of which were iron and the end-plates magnetite. Since the magnetic inversion-point of magnetite is much lower than that of iron, the magnetic circuit, when the transformer coil was heated, would be broken when the inversion-

Fig. 2.

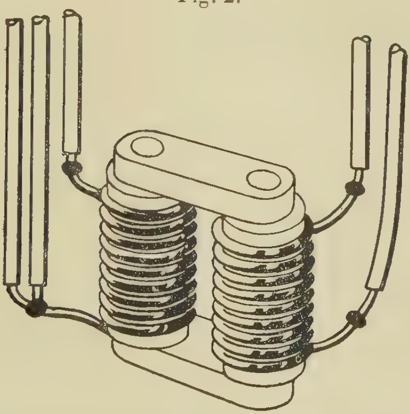


Diagram of the induction-unit (enlarged). Primary and secondary coils, insulated by silica glass, are wound on a frame made of the material under investigation. Height, 1 cm.

temperature of the magnetite had been attained, and the magnetic behaviour would therefore be practically the same as if the core were made entirely of magnetite.

The primary and secondary coils of the transformer, or induction-unit, were wound on thin-walled silica-glass tubing, and were slipped over the round side-bars. The primary winding consisted of nine turns of 0·4 mm. bare platinum wire wound with 0·3 mm. space between the turns. The secondary winding was made to serve the purpose also of two thermocouples for measuring the temperature at the top and bottom of the sample. This was accomplished as follows :—The secondary winding

was of 10 per cent. platinrhodium 0·4 mm. in diameter, with the same number of turns as in the primary, and at the top was fused to a wire of pure platinum 0·4 mm. in diameter, thus forming a thermocouple-junction for measuring the temperature of the upper end of the induction unit. At the lower end the platinrhodium wire was extended beyond the winding to form the second terminal of the coil, and there was fused to it, near the lower end of the coil, another wire of pure platinum forming a second thermocouple-junction. The three terminal wires were connected to wires leading through the top end of the bomb, thence through cold-junctions insulated by glass tubes and dipping in ice, and thence to a switching arrangement for connecting to the device for measuring the inductive effect or to the potentiometer, on which could be measured the temperature of either the top or the bottom of the induction unit or the difference between them.

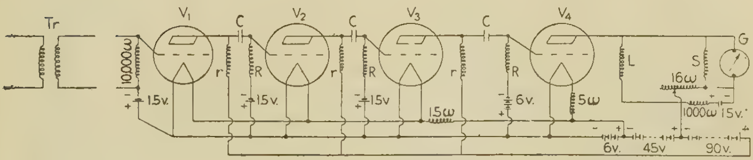
The limitation of size due to the small internal diameter of the furnace and the necessity for electrical insulation of a character which will withstand high temperatures made the determination of the magnetic induction somewhat more difficult than usual.

In the preliminary work the inductive effect was measured with the potentioneter galvanometer, using it as a ballistic galvanometer. The primary coil was connected to a storage-cell and a rheostat. Upon making and breaking the circuit, the ensuing galvanometer deflexion was a measure of the magnetic induction within the specimen or of its permeability, and the absence of a ballistic throw indicated that the Curie-point had been reached. Although this method was moderately satisfactory with a current of about 1·3 amperes in the primary coil, it was not sensitive enough for best results, particularly with the materials of lower permeability than that of iron, and it was supplanted by a much more sensitive method. By this method alternating current (at 60 cycles/sec.) was supplied to the primary coil, and the electromotive force inductively generated in the secondary was amplified by an electron-tube amplifier to which was connected a tube rectifier and, finally, a portable galvanometer. With only 0·3 ampere in the primary coil, this method gave ample sensitivity with all the material investigated, and, furthermore, allowed readings to be taken rapidly and with ease and certainty. So long as the specimen was ferro-

magnetic the galvanometer showed a steady deflexion, the magnitude of which depended on the permeability of the specimen. At or above the inversion temperature the deflexion was zero.

The wiring diagram of the electrical arrangement for measuring the inductive effect is shown in fig. 3. The secondary of the transformer was connected to the grid-filament circuit of the first tube of a three-stage resistance-coupled amplifier the output of which was connected to the rectifying tube V_4 of the well-known "201-A" type. This tube has a negative grid-bias sufficient to reduce the plate-current to a small value (about 30 microamperes). The portable galvanometer, or microammeter, G, is connected in the plate-circuit of V_4 in such a way that the residual plate-current may be balanced out by adjustment of a small rheostat, and that the microammeter reads

Fig. 3.



The electrical circuit of the three-stage electron-tube amplifier and rectifier for reading the inductive effect as a steady deflexion on a direct-current instrument. The coupling condensers C have a capacity of 0·1 microfarad, the resistances *r* are 0·5 megohm each, and the grid-leaks R 5 megohms each.

zero when there is no alternating-current input. The static amplification constant of the tubes V_1, V_2, V_3 is about 20 and the overall voltage amplification between the secondary of the induction-unit and the grid of the tube V_4 is about 3500, even at frequencies as low as 60 cycles per second. The scale of the microammeter consisted of 180 divisions, one division being equivalent to 0·33 microampere, and the aggregate sensitivity of the whole apparatus was such that an input of one-half millivolt gave full-scale deflexion on the galvanometer. Somewhat greater sensitivity could have been obtained by the use of the grid-leak condenser method of rectification instead of the biased grid method here employed, but the latter gave satisfactory sensitivity, and with that

method the plate-current, which was small, could readily be balanced out. It should be noted that with practically all electron-tube rectifiers the output varies approximately as the square of the input.

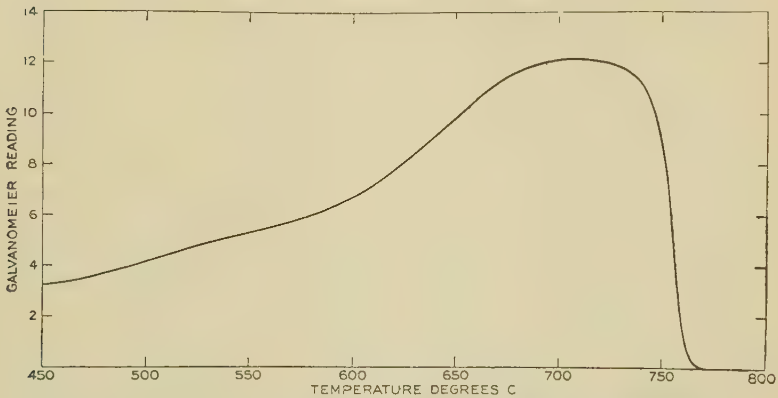
The experimental procedure was as follows. The induction-unit having been placed in the silica-glass tube and packed with crushed silica-glass, and the five terminals having been joined by fusion to the appropriate wires on the under side of the top lid, the lid was carefully lowered on the bomb and about 50 tons applied with the hydraulic press. Liquid carbon dioxide was then pumped into the bomb until the desired pressure was attained, while at the same time the force exerted by the press was increased. Ordinarily it was not necessary to raise the pressure to the required amount entirely by pumping, since the increase of pressure during the heating would supply a considerable part of the pressure. Thus, for example, it was sufficient to start with a pressure of 1200 atmospheres in order to have 2000 atm. when the temperature had reached 260°, and with a pressure of only 600 atm. to have 2000 atm. at 800°.

During the first part of the heating the rate was fairly rapid, but at a temperature about 200° below the inversion-point the rate was lowered to 10 or 15° per minute, and the series of readings was started. There were recorded: primarily, (1) the inductive effect as indicated by the galvanometer, (2) the temperature of one end of the induction-unit, (3) the difference in temperature of the two ends, (4) the pressure; and, secondarily, the current through the primary of the induction-unit and the current through the furnace-winding. Initially, readings were taken at intervals of about 10°, but as the temperature of the inversion was approached the heating-rate was decreased to about 1° per minute and readings were taken at intervals of 1° or less. When the critical temperature had been passed, as shown by the inductive effect having fallen to zero, the heating-current was reduced so that the temperature fell slowly, and a cooling-curve was determined in the same manner as the heating-curve. The pressure was then lowered somewhat by allowing carbon dioxide to escape through a valve, and another heating- and cooling-curve obtained at the lower pressure. In this way observations at several pressures were made with one filling of the bomb.

Experimental Results.

A typical heating-curve for iron is shown in fig. 4, at which the temperature is plotted as abscissa and the electromagnetic effect as ordinate. This effect is given in arbitrary units, being merely the reading of the galvanometer, which, as explained above, indicates the increase in plate-current of the last electron-tube in the amplifier-rectifier, and is approximately proportional to the square of the e.m.f. generated in the secondary of the transformer-unit, and hence approximately to the square of B the magnetic induction, or the number of magnetic lines of force threading the secondary coil. The field-

Fig. 4.



Typical heating-curve for iron, The magnetic induction increases with rising temperature and then drops sharply at the Curie-point. The cooling-curve is practically coincident with the heating-curve for iron, and also for nickel, magnetite, and nickel steel, but not for meteoric iron (see fig. 5).

strength may be calculated from the relation $H = 4\pi in/10l$, in which H is the field-strength in gauss, that is, in lines per cm.[2], i the current in amperes, l the length of the magnetic circuit in cm., and n the number of turns in the primary. The current was usually about 0·3 ampere, and the other constants are given above in connexion with the description of the transformer-unit. From this H turns out to be 1·6 gauss (average), which is rather low, being about eight times the mean horizontal intensity of the earth's magnetic field. Unless there is magnetic leakage,

the value of B in that part of the core within the secondary will equal μ H, μ being the permeability, but at temperatures near the critical temperatures the permeability drops to practical unity, and the magnetic induction B is then so low that the galvanometer in the output of the amplifier reads zero.

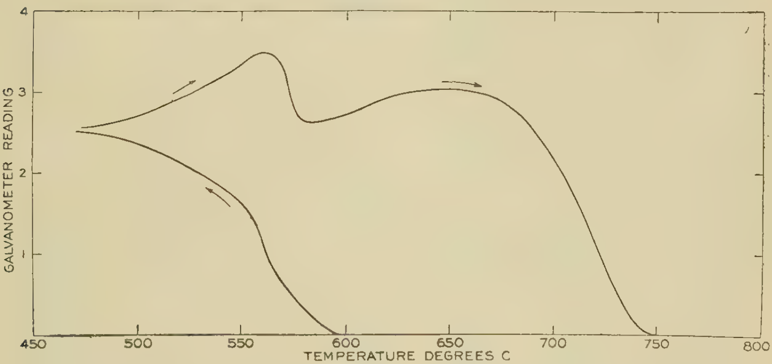
From fig. 4 it may be seen that the inductive effect at first rises with increasing temperature. This apparently represents the normal behaviour of the permeability of ferromagnetic materials in weak fields. The rise continues to a point just below the inversion-point where the curve begins to drop sharply, tailing off a trifle just before it falls to zero. This latter effect is probably due to two causes: first, the lack of sharpness in the inversion itself, and second, the temperature-difference within the specimen. Such differences in temperature obviously will spread out the inversion over an interval (when the temperature is read on either thermocouple) approximately equal to the difference in temperature of the two ends of the specimen. But the change of permeability near the critical temperature is very rapid, and it may be seen from the construction of the induction-unit that when either end of the core is above the transformation-point, there will be a marked break in the magnetic circuit. The magnetic flux which threads the secondary winding will then fall to a very low value, and on account of the characteristics of the amplifier-detector will give a zero-reading. The proper temperature to record, therefore, is that of the *hotter* end, and this was done in all cases.

The same type of heating-curve was obtained for nickel, magnetite, and nickel steel as for iron. Moreover, for these four materials the cooling-curve was practically coincident with the heating-curve. For the meteoric iron, on the other hand, a different type of curve was obtained, as shown in fig. 5. Here, with increasing temperature, the permeability rises to a maximum at 550°, falls rapidly, then begins to rise again, and finally falls rather slowly to the transformation-point, 745°. Upon being cooled, the material does not begin to regain its ferromagnetism until a temperature of 600° is reached. Meteoric iron is a complicated mixture of iron, nickel, and other substances, and it is evident from the heating- and cooling-curves that there are present at least two separate phases of different magnetic properties. The meteoric

iron is essentially a nickel steel with about 7 per cent. nickel. It is of interest to note that nickel steels with low nickel content are known to regain their magnetism at a temperature considerably lower than that at which they lose it upon being heated.

Although many interesting and unexpected problems were presented in the course of this work, the object of the present investigation was to determine the magnetic inversion-temperatures under various pressures, and hence it was only the end-points of the curves that were of primary interest. The curves did not fall very sharply to zero, yet it was possible in most cases to determine, with an error of less than 1°, the temperature at which the

Fig. 5.



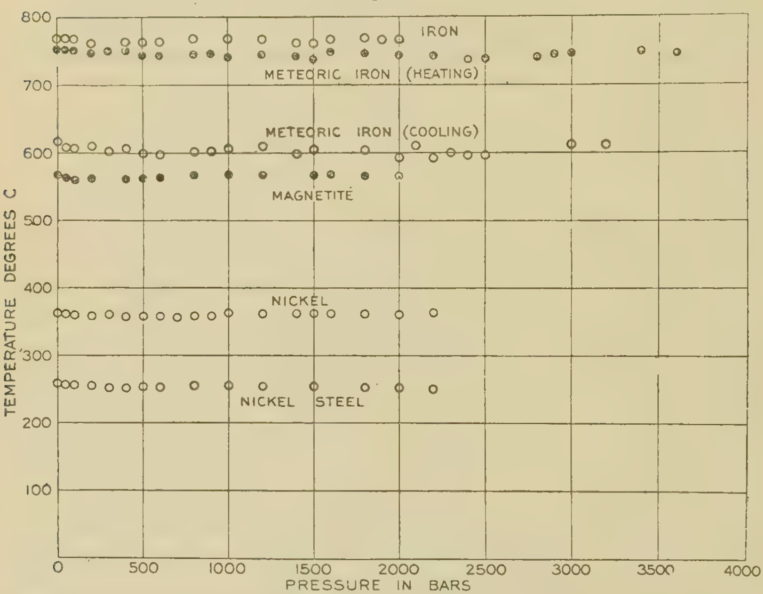
Typical heating- and cooling-curves for meteoric iron. The ferromagnetism is lost on heating at a considerably higher temperature than that at which it is regained on cooling.

induction effect became zero. The transformation-points are collected together in Table I. and are plotted in fig 6. Altogether about 600 separate runs were made, each including a heating- and cooling-curve. When more than one determination was made at a given pressure the results have been averaged to give a single point. The heating- and cooling-points also were lumped together, except for meteoric iron.

The values as given have been corrected for a small error which was not discovered until after all the measurements had been made. The original uncorrected temperatures of inversion dropped sharply with increase of

pressure, reaching a minimum at 500 to 800 bars, and then rose slightly for the remaining pressure-range. This effect was very puzzling, more especially because the same type of curve was obtained for each of the substances investigated. Subsequently Dr. R. E. Gibson, using the same apparatus, made some measurements on the effect of pressure on the α–β inversion in quartz and on the melting-points of certain metals and salts. These seemed to show an anomaly in the temperature, particularly

Fig. 6.



The temperatures of the magnetic inversion (Curie-points) for the five ferromagnetic materials under investigation as a function of pressure. The pressure-coefficient is practically zero for all of the materials.

in the pressure-range from 1 to 800 bars. Suspicion was then directed towards the thermocouple. The platinum and platinrhodium wires are so soft that they are likely to pinch off in the insulating packing in the bomb-lid, either in the process of assimilating the packings or, later on, when the bomb is subjected to pressure. To avoid this difficulty short lengths of iron and nichrome wire (not chromel) had been substituted for those parts of the platinum and platinrhodium wires, respectively, that passed

through the packing material. Since an iron-nichrome thermocouple at low temperatures has almost exactly the same thermoelectric power as the platinum-platin-

TABLE I.

Summary of Corrected Results for the Effect of Pressure on the Magnetic Inversion-point.

Pressure (bars).	Critical temperature of magnetization (Curie-point).					
	Nickel Steel.	Nickel.	Magnetite.	Meteoric Iron.		Iron.
				Heating.	Cooling.	
1	259	362	569	619	754	768
50	258	361	565	608	752	769
100	257	360	561	607	752	768
200	255	359	563	610	747	762
300	252	360	...	602	750	
400	251	357	561	608	750	763
500	254	358	564	598	745	764
600	253	358	564	596	745	764
700	...	356				
800	255	359	568	602	745	767
900	...	359	...	604	746	
1000	255	362	568	607	742	768
1200	254	361	568	610	745	767
1400	...	362	...	599	742	762
1500	254	362	567	605	738	761
1600	...	362	568	...	750	767
1800	253	361	567	604	746	769
1900	...	...	...	...	...	766
2000	252	361	567	593	744	766
2100	...	...	...	611		
2200	...	361	...	593	744	
2300	...	...	...	600		
2400	...	...	...	597	738	
2500	...	...	...	597	738	
2600	253					
2800	...	...	...	...	741	
2900	...	...	...	...	745	
3000	...	...	...	611	747	
3200	...	...	...	612		
3400	...	...	...	...	748	
3600	...	...	...	...	747	

rhodium thermocouple, the introduction of the extraneous material into the thermoelectric circuit should make no change in the thermoelectromotive force provided that each pair of junctions were at the same temperature. It is believed that in constructing the bomb a mistake had been made in the kind of wire that was put into one of the

packings, because later measurements on the α–β quartz inversion and on certain melting-points made with a thermocouple consisting entirely of chromel and alumel showed no anomalies. Moreover, from Dr. Gibson's two sets of measurements it was possible to construct a table of corrections for the various pressures and temperatures. These corrections are shown in Table II.

It may be noted in passing that the effect of pressure on the thermoelectromotive force is probably not very large for the platinum-platinrhodium thermocouple. The thermoelectric behaviour of platinum under pressure has

Table II.

Temperature Corrections as determined by Subsequent Measurements on Various Transformations.

Pressure (bars).	Additive correction for given temperature (t).				
	t=260.	t=360.	t=570.	t=620.	t=760.
100	1	1	2	2	2
200	2	3	4	4	5
300	3	4	6	7	8
400	4	5	8	9	10
600	5	6	10	11	13
800	6	8	12	13	16
1000	6	8	12	13	16
1500	5	7	11	12	14
2000 and higher	5	6	10	11	13

been measured by Bridgman *, but no results are available for platinrhodium. Probably the pressure-correction for the platinum-platinrhodium couple is of the same order of magnitude as for copper-constantan, and can safely be neglected in the present work. For the copper-constantan couple the effect of 1000 bars pressure on the hot junction is to lower the indicated temperature by about 0·06° for each 100° difference in temperature of the two junctions.

Discussion.

From the inversion-temperatures at various pressures as plotted in fig. 6 it may be seen that there is no very pronounced trend of the transformation-points with increasing pressure. The results are not as regular as

* P. W. Bridgman, Proc. Acad. Arts Sci. liii. pp. 269-386 (1918).

might have been expected, especially for the meteoric iron, both heating and cooling; but this material is complex in composition, and apparently the inversion-temperature depends to some extent on the previous history of the sample. Nickel and nickel steel gave the most regular results. This is probably because, at the low temperature of the transformations in these two materials, the temperature was much more uniform within the bomb.

Pressure seems to have little or no effect on the magnetic inversion-points for the five ferromagnetic materials under investigation. If there is any trend at all, it is in the nature of a slight decrease in temperature with increasing pressure. It is reasonable to suppose that this same situation holds at very much higher pressures, and at even the extreme pressures deep within the earth. Therefore, since the temperature in the interior cannot be less than several thousand degrees, and since the effect of pressure is *not* to raise the magnetic inversion-point, it can be stated with almost complete assurance that the metallic core of the earth does not have the familiar magnetic quality of iron or nickel-iron alloys, and that the iron core is without influence on the earth's magnetic field. There is, of course, a possibility that pressures of more than a million atmospheres may produce unexpected effects analogous to the well-known phenomenon of supra-conductivity, which is encountered at extremely low temperatures. It has been suggested that such pressures may cause a complete breakdown of the whole atomic structure with profound and unpredictable consequences. Such considerations, it is true, introduce an element of uncertainty into conclusions of any kind regarding matter under extreme temperatures and pressures; but, whatever may be the situation in the interior of stars, no evidence, either direct or indirect, has yet been obtained which makes it appear probable that anything very revolutionary will take place under the conditions of temperature and pressure in the interior of a planet like the earth. On the whole it seems a fair inference that the pressure-coefficient of the magnetic inversion-point remains zero or negative even at very high pressures in the earth's interior, and that, consequently, the permeability of the iron core of the earth is not significantly higher than that of ordinary rocks.

From the purely physical viewpoint the measurements here recorded are particularly interesting because they

demonstrate that the transformation from the magnetic to the non-magnetic state takes place without volume-change, or at most with a very minute one. It seems to be the current belief that there is an appreciable change in volume of iron, for example, although an examination of the literature does not reveal any very definite values. Many of the measurements have been on the linear expansion of rods and wires, and do not necessarily determine the change in volume. Now the increase in volume, Δv, in any invariant transformation is connected with the pressure-coefficient dt/dp and the heat absorbed, Δh, by the Clausius-Clapeyron equation, $dt/dp = T\Delta v/\Delta h$, in which T is the absolute temperature. From Table III., in which certain data concerning the magnetic transformation in the five materials investigated have been collected, it may be seen that Δh for nickel is 0·6 decijoule per gram and T is 633 ; from fig. 6 it appears that the variation of t with p for nickel is probably not numerically greater than $-0{\cdot}001$ deg./bar. Hence it follows that Δv is equal to or less (numerically) than -1×10^{-4} cm.^{3}/g. Similarly for iron, if the limiting value of dt/dp be taken as $-0{\cdot}002$, it can be calculated that Δv is -5×10^{-4} cm.^{3}/g or less (numerically). The calculation cannot be made for the other three materials because values of Δh are not available, but it is a fair inference that for them the same situation would hold.

It appears, then, that the volume-change during the magnetic transformation is zero, or very small. This may possibly be expected from the fact that, according to X-ray analyses, the atomic arrangement of the magnetic and non-magnetic iron (α and β) is identical, but although much progress has been made in determining the interior arrangement of atoms, the most recent theories of the atom and of the ultimate cause of magnetism do not shed much light on the question of the volume-change at the magnetic inversion-point.

Summary.

Measurements have been made on the effect of pressure on the magnetic inversion-temperature (Curie-point) for five ferromagnetic materials—iron, nickel, magnetite, nickel steel (35 per cent. nickel), and meteoric iron from the Canyon Diablo meteoite.

On account of the limitations of space within the pressure-apparatus, and for other reasons, it was necessary to devise a method in which the complete induction-unit occupied a volume less than one-half a cubic centimetre and was adequately insulated even at temperatures of about 1000°. The materials, in the form of small transformers, were subjected to pressure in a water-jacketed bomb with internal electric heating, and the primary coil was supplied with 60-cycle current, the output of the secondary being stepped up with an electron-

TABLE III.

Miscellaneous Data concerning the Five Ferromagnetic Materials under Investigation.

Material.	Composition.	Curie-point. Previous values. Range.	Curie-point. Previous values. Weighted mean.	Curie-point. Present value.	Heat of inversion, cal./g.
Iron	Fe	740–810	768	768	5_5
Nickel ...	Ni	349–380	360	362	1_2
Magnetite...	Fe_3O_4	530–590	575	569	
Nickel steel (35 %).	Fe–Ni	261		259	
Meteoric iron (Canyon Diablo).	Essentially Fe with 8% Ni.	...		754 (heating). 619 (cooling).	

tube amplifier, rectified, and then read on a portable microammeter. When the specimen was heated, the deflexion of the instrument dropped to zero at the inversion-temperature. Pressures as high as 3600 bars (metric atmospheres) were used.

Although the results show minor inconsistencies, they demonstrate that the effect of pressure on the inversion-point is practically nil, but the possibility of a slight decrease in temperature with increasing pressure is not excluded. From this it follows, since the energy-change is of considerable magnitude, that the volume-change accompanying the magnetic transformation is zero, or very small.

If, as seems reasonable, this tendency still holds qualitatively at the very high pressure in the interior of the earth,

it must be concluded that the nickel-iron core, which has a diameter of about one-half and a volume about one-ninth of the earth's, is at a temperature far above that at which it could be magnetic. The nickel-iron core, therefore, in spite of its great volume, has no important direct influence on the earth's magnetic field.

Carnegie Institution of Washington:
Geophysical Laboratory (L. H. Adams).
Department of Terrestrial Magnetism (J. W. Green).
January 1931.

XXX. *Striated Discharges.*

By L. G. H. Huxley, *M.A., D.Sc.**

1. STRIATED discharges have been the subject of many experimental investigations, and various theories have been suggested in order to explain this form of the positive column in discharge-tubes. In most cases some special hypothesis has been brought into the theory, and the statistical distribution of the energies of the electrons has been disregarded. The advocates of these theories maintain that very few electrons attain energies much greater than a certain critical potential, when the electrons move in a field where the electric force Z is small compared with the pressure p of the gas—for example, in helium, when the ratio Z/p is less than 5 (Z being expressed in volts per centimetre and p in millimetres of mercury).

In some gases this critical potential is said to be the lowest metastable or resonance potential, and in others it is said to be the ionizing potential. It is stated that each electron goes through a cyclic determinate process in which its energy increases slowly as it moves in the direction of the electric force until it acquires the energy corresponding to the critical potential. The electron then loses its energy suddenly in a collision with an atom of the gas. In accordance with this hypothesis it is maintained that in the striated positive column the difference of potential between two striations is equal to the critical potential.

2. There are several fallacies in these theories which have already been pointed out, and it is easily seen that they do not provide a rational explanation either of the uniform

* Communicated by Prof. J. S. Townsend, F.R.S.

positive column in discharges or of the striated positive column.

There is no cycle depending on a critical potential which is approximately the same for any considerable number of electrons. When electrons start from rest they acquire kinetic energy by moving in the direction z of the force and lose or gain energy in small amounts in collisions with atoms, the losses being on an average greater than the gains. These small interchanges of energy are different for different electrons, so that electrons move through different distances z before they acquire the energy corresponding to the first critical potential. When electrons have acquired that amount of energy, many of them continue to gain energy by moving in the direction of the electric force, and very few lose the amount of energy corresponding to the first critical potential. In order to explain the radiation from the discharge it is necessary to assume that, even when the ratio Z/p is very small, many electrons acquire energies which are greater than the amount lost in the collisions which excite radiation of one particular wave-length.

3. Some experiments which I have made to determine the electric force in the positive columns of direct current discharges in neon are of interest in considering an explanation of the striated discharges.

The apparatus consisted of a pyrex tube 50 centimetres long and 1·7 centimetres in internal diameter, containing movable electrodes.

Each electrode consisted of two aluminium disks fixed at their centres to the ends of an aluminium rod 3 centimetres in length.

Short pieces of soft iron were fixed to the electrodes so that the distance between them could be adjusted by a magnet.

Electrical connexions were made to the electrodes by flexible spirals of thin copper wire joined to molybdenum wires sealed into the ends of the pyrex tube. A direct current was maintained in the gas by a battery of small accumulators, and the electric force in the positive column was measured by finding the increase in the potential between the electrodes, required to maintain a given current, when the distance between the electrodes is increased by a measured amount.

The current was regulated by a thermionic valve in series with the battery of accumulators.

The neon was purified by storage over charcoal cooled

by liquid air, in the manner which had been successful in other experiments *. The discharge-tube was heated with a blow-pipe while washing it out with pure neon, so that the impurities given off from the electrodes and the pyrex tube were removed with the neon. When this operation had been carried out for several days, impurities were no longer given off when the tube was strongly heated and the spectrum of the gas in the tube was that of pure neon.

Measurements were then made of the electric force in the positive column at various pressures, with a current of 2 milliamperes.

The following results were obtained :—

Pressure in millimetres of mercury.	Force in volts per centimetre.
5	5·4
1	4·6
·6	3
·4	2

At pressures less than 1 millimetre the positive column was striated, so that for these pressures the forces given in the table are the average values of the electric force.

The particular case of the positive column at a pressure of ·6 millimetre may be considered.

The striations were regularly spaced with their centres 3·6 centimetres apart, so that the difference of potential between the centres of adjacent striations was 11 volts. There is an appreciable difference between this potential and the potential 16·4 volts which is said to be the first critical potential of neon.

As there is no abrupt change in the average value of the electric force when the positive column becomes striated, it is reasonable to suppose that both striated and unstriated discharges may be explained on the same principles.

4. In the uniform positive column the conductivity is maintained in gases at high pressure by comparatively small electric forces. Thus in neon the force is 5·4 volts per

* Huxley, Phil. Mag. v. p. 721 (1928).

centimetre when the gas is at 5 millimetres pressure in a tube 1·7 cm. in diameter. The high conductivity, as has been explained by Townsend, is due to a positive charge in the gas. According to his theory the current is maintained when the rate at which the positive ions and the electrons flow to the sides of the tube is balanced by the rate at which new ions are generated by the process of ionization by collision. Since the electrons diffuse more rapidly than the positive ions, a positive charge accumulates in the gas which tends to reduce the flow of the electrons and to increase the flow of the positive ions towards the surface of the tube. A steady stage is reached and a constant current is maintained in the tube when the positive ions and the electrons flow at the same rate to the sides, and the loss of ions and electrons due to this effect is balanced by the rate at which they are generated in the gas by the process of ionization by collision *.

In the steady motion the relation between the coefficient of ionization α, the coefficient of diffusion K_1 of the electrons, and the velocities W_1 and W_2 of the electrons and positive ions reduces to the equation †

$$\alpha W_1^2/K_1W_2 = C^2,$$

where C is a constant depending on the concentration of electrons at the surface of the tube and on the diameter of the tube.

The effect of the positive charge in the gas is the same as if the rate of diffusion K_1 of the electrons were reduced to K_1W_2/W_1, and there is a corresponding reduction in the rate of ionization α required to maintain the current.

5. The theory of the striated positive column is more complicated. From the distribution of the light in the tube it may be seen that the energy of agitation of the electrons varies along the axis of the tube. There are also corresponding variations in the rate of diffusion K_1, in the velocities W_1 and W_2, and in the number of electrons n per c.c. of the gas. As a first approximation the differential equation for n in terms of the distance z along the axis, and the distance r along a radius, may be expressed in the form

$$\frac{d^2n}{dr^2} + \frac{1}{r}\frac{dn}{dr} + \frac{d^2n}{dz^2} + \frac{\alpha W_1^2 n}{K_1W_2} = 0.$$

* J. S. Townsend, 'Electricity in Gases,' Section 302, Clarendon Press, Oxford (1915).

† J. S. Townsend, *Comptes Rendus,* clxxxvi. p. 55 (9th Jan. 1928).

The coefficient of the last term may be taken as a constant, the values of α, W_1, W_2, and K_1 being the mean values of these quantities.

The solution of this equation may be expressed in the form

$$n = AJ_0(rc) + BJ_0(r\sqrt{c^2-k^2}) \sin kz,$$

where $2\pi/k$ is the distance between the striations and A and B are arbitrary constants.

Theoretically k may have any value, and there is no evidence to show that the potential difference between two striations should be equal to a critical potential.

The experiments show that the amplitude B tends to increase with respect to A under certain conditions, as the striated form becomes more pronounced.

Thus the theory is consistent with the phenomena, although it does not indicate the value of Z/p where the coefficient B increases with respect to A.

XXXI. *The Conductivity of Gases in Uniform Electric Fields.*

By S. P. McCallum, *M.A., D.Phil., Fellow of New College, Oxford, and* F. Llewellyn Jones, *B.A., Senior Demy of Magdalen College, Oxford* *.

1. In many recent publications various theories of electrical discharges have been elaborated which are based on hypotheses of doubtful validity. They include several laws relating to the loss of energy of electrons in collisions with atoms and laws relating to the properties of metastable atoms. These laws may be convenient in order to account for the results of experiments on thermionic currents in gases at low pressures, but they fail to give a reasonable explanation of several of the principal properties of discharges in gases at high pressures.

In this paper we propose to draw attention to objections that have already been raised to these theories, and to compare the results to which they lead with the results of experiments on monatomic gases which have recently been made at the Electrical Laboratory, Oxford.

2. In discharges through monatomic gases at high pressures it has been found that the electric force in the uniform positive column is comparatively small. If Z be

* Communicated by Prof. J. S. Townsend, F.R.S.

the force expressed in volts per centimetre, and p the pressure in millimetres of mercury, the ratio Z/p is less than 2 in the positive column of discharges in helium * in tubes about 3 centimetres in diameter when the gas pressure is greater than 6 millimetres. The mean energy of agitation of the electrons in helium corresponding to this value of Z/p is 4 volts. Similarly in neon † when the pressure exceeds 15 millimetres; the ratio Z/p is less than ·5, and the mean energy of agitation of the electrons corresponding to this value of Z/p in neon is 5·2 volts. Also it was found that with small currents the electric force is independent of the current, which shows that the numbers of electrons and positive ions per cubic centimetre of the gas are proportional to the current. The gases in these experiments contained no impurities that could be detected by examining the currents in the wide tubes with a direct-vision spectroscope.

3. In order to explain the conductivity of gases at high pressures some physicists maintain that the ionization is due indirectly to metastable atoms. According to the laws they have adopted these atoms are formed by the collisions of electrons with normal atoms of the gas when the electrons have energies greater than a certain critical value which may be expressed as a potential. In helium this critical potential is about 20 volts and in neon about 16·5 volts. These potentials are the lowest or first critical potentials, but there are several other potentials greater than these which are less than the ionizing potential, the latter being about 5 volts greater than the first critical potential.

When electrons have energies less than the first critical potential it is stated that they lose a very small proportion of their energy in a collision with an atom, but electrons that have larger energies may lose the amount of energy corresponding to any of the critical potentials. Thus, when electrons acquire energy by moving in the direction of the electric force there is an abrupt change of energy due to collisions with atoms after the electrons have acquired the amount of energy corresponding to the first critical potential. The probability of the electrons losing their energy in these collisions is said to be high; so that the number of electrons that acquire energies greater than the first critical potential is negligible in gases at high pressures, since the electrons collide with a large number of

* F. Llewellyn Jones, Phil. Mag. xi. p. 163 (Jan. 1931).

† P. Johnson, Phil. Mag. x. p. 921 (Nov. 1930).

atoms while moving a short distance in the direction of the electric force.

4. It is asserted that metastable atoms have two distinct properties which account for the conductivity under the conditions described in the preceeding section.

The metastable atom is said to have the power of transferring its energy to an electron, so that in a collision with a metastable atom an electron has its energy increased by the amount corresponding to a critical potential. Thus the normal atoms of monatomic gases are supposed to have the property of absorbing all the energy of some electrons and transferring it to others. In this way an electron in helium moving with an energy corresponding to 5 volts may have its energy increased to 25 volts. It thus has sufficient energy to ionize an atom of helium. The rate at which new ions may be formed by this process is proportional to the product of the number of metastable atoms and the number of electrons. With photoelectric currents between parallel plates the number of metastable atoms is proportional to the current, so that on this hypothesis the rate at which new ions may be formed is proportional to the square of the current. It is well known that this result is contrary to the experiments on the rate of increase of currents due to the motion of electrons in a uniform electric field.

It has always been found that the coefficient of ionization obtained from experiments with photoelectric currents is of the same order as that in the positive column of a gas, so that it may be supposed that in both cases the phenomena are due to direct impact of electrons with normal atoms of the gas.

5. It has also been asserted that the metastable atoms also form new ions by ionizing molecules of other gases which may be present in the form of impurities, provided that the ionizing potential of these molecules is less than the energy of the first critical potential of the monatomic gas. Consequently it has been suggested that the conductivity of gases at high pressures may be explained by this process of ionization, and that the amount of impurity required to maintain the conductivity of the gas is so small that it cannot be detected spectroscopically. This might be the case if the current were very small, but it is impossible to admit that currents of the order of 10 or 100 milliamperes are due to impurities which cannot easily be observed spectroscopically. In the direct-current discharge in a long tube the impurities in monatomic gases are swept out of the uniform positive column and carried in the form of positive ions to the negative electrode, where they

may be absorbed by charcoal at the temperature of liquid air. In order to maintain the current it would be necessary to have a continuous supply of impurity proportional to the current throughout the positive column, and it is easy to estimate the amount of impurity required to account for the conductivity on this hypothesis. In the positive column of a direct-current discharge through a gas at a few millimetres pressure in a wide tube the numbers of positive ions and electrons are approximately the same, and the velocity of the electrons is about 100 times that of the positive ions. A simple calculation shows that with a current of 100 milliamperes the positive ions would transfer across any section of the tube in one minute an amount of gas which would fill a volume of 10 cubic centimetres to a pressure of about half a millimetre. It is unreasonable to suppose that such an amount of gas could be in a tube without being observed spectroscopically.

6. It is thus seen that there are fundamental objections to these theories of conductivity depending upon the collisions of metastable atoms, and the following considerations show that there are other objections to the theories developed on similar lines relating to the positive column of currents in monatomic gases.

In a recent paper on this subject Headrick and Duffendack* investigate the relation of the force Z to the pressure p in the positive column of the discharge through monatomic gases. They adopt a theory of the discharge due to Morse † and Schottky ‡, where the force is expressed by an equation involving a critical potential. Morse, Headrick and Duffendack agree that this potential in helium is the first critical potential (19·77) volts, and that, according to the theory they adopt, the force in the positive column is directly proportional to the pressure.

It may be seen from ordinary considerations that this conclusion is directly opposed to simple observations of the light from the discharge. If the force Z were proportional to the pressure p the ratio Z/p would be constant, and since the velocity W of the electrons in the direction of the electric force depends only on Z/p, this velocity would be the same at different pressures, so that the total number N of electrons per c.c. of the gas would also be the same at different

* L. B. Headrick and O. S. Duffendack, Phys. Rev. xxxvii. p. 736 (March 15, 1931).

† P. M. Morse, Phys. Rev. xxxi. p. 1003 (June 1928).

‡ W. Schottky, *Phys. Zeits.* xxv. p. 342 (July 1924).

pressures with a given current (NeW). In addition it has been found experimentally * that the mean energy of agitation E_1 of the electrons is a function of Z/p, so that E_1 would also be constant at all pressures.

The energies E of the electrons are distributed about the mean energy E_1, and the proportion N_x/N of the total number of electrons with energies greater than any value E_x would also be the same at all pressures.

Thus with a given current the number of collisions made by electrons with energies between any two limits E_x and E_y with atoms of the gas would be proportional to the number of atoms per c.c., and therefore proportional to the pressure. Since the light is presumably caused by effects of collisions with atoms of the gas in which the electrons have energies exceeding a certain value E_x, this theory leads to the conclusion that the intensity of the light should increase with the pressure when the current is maintained constant.

It may be seen by simple observations that in helium the intensity of the light diminishes as the pressure is increased. In fact, the experiments † which have been made to measure the change of intensity with the pressure show that the intensity of the light in the visible spectrum of helium is approximately inversely proportional to the pressure.

Thus any theory which leads to the conclusion that the force is proportional to the pressure in the uniform positive column must be considered to be inconsistent with the principal properties of the currents.

7. A theory of the striated positive column has also been given which is based on the same assumption that there is a high probability that the energy of an electron is transferred to an atom in a collision where the energy of the electron exceeds a certain critical value. In the first form of this theory, which was suggested by Holst and Oesterhuis ‡, it was maintained that this critical value was the ionizing potential, and that the potential difference between two striations was approximately equal to this ionizing potential. In a later form of the theory it is assumed that in monatomic gases it is the first critical potential which is of importance §. In fact it is contrary to the laws of collisions to suppose that in

* J. S. Townsend, 'Motion of Electrons in Gases' (Clarendon Press, Oxford); also 'Journal of the Franklin Institute,' vol. 200 (Nov. 1925).

† J. S. Townsend and F. Llewellyn Jones, Phil. Mag. xi. p. 679 (March 1931).

‡ G. Holst and E. Oesterhuis, Phil. Mag. p. 9 (Dec. 1923); *Comptes Rendus*, clxxv. p. 577 (1922).

§ P. M. Morse, *loc. cit.*

gases at high pressures electrons acquire the amount of energy necessary to ionize an atom, since they lose their energy in collisions with atoms when they attain the amount corresponding to the first critical potential.

If this theory be accepted it would be necessary to assume that there are critical potentials in helium less than 19·77 volts, and in neon less than 16·5 volts, since striations have been observed in direct-current discharges through these gases where the potential difference between the striations was less than 12 volts. Every precaution was taken to purify the gases, and no lines due to impurities were observed in the spectrum of the light from the striated column. If there were small traces of impurities they would have been carried to the cathode by the action of the current. But there was no change observed in the potential difference between striations due to the current being maintained for a long time. Thus it would be unreasonable to contend that the striations are due to impurities.

Argon. Diameter of Tube 2 cm.

p.	Z.	*d*.	λ.	V*s*.
·74	3·02	2·7	220	8·2
2·98	4·62	2·0	220	9·0
2·98	4·56	2·0	80	9·0
5·48	7·8	2·0	220	15·6
8·9	11·5	2·0	220	23·0
12·6	14·8	2·0	220	29·0

8. This theory of striations cannot be reconciled with the fact that striations similar to those obtained with direct currents are also obtained with high-frequency currents. In these currents the electrons move about a mean position, and in each half-period of the oscillation the mean distance z they move in the direction of the electric force Z is small, and the potential $z \times Z$ may be about one or two volts.

In argon striations are obtained with high-frequency currents over a wide range of gas pressures and over a wide range of frequencies. In the following table the potential difference V*s* between successive striations is given in volts, the average force Z along the tube in volts per cm., the distance *d* between the striations in cm., the wave-length λ of the oscillation in metres, and *p* the pressure of the gas in mm. of mercury.

The potential V_s is almost independent of the frequency, with oscillations from 40 metres to 220 metres in wave-length, and V_s changes from 8·2 volts to 29 volts as the pressure is changed from ·74 to 12·6 mm.

It will also be seen that the distance d is almost the same at all pressures from 3 mm. to 12·6 mm.

The mean displacement of the electrons in the direction of the axis due to the electric force $Z \sin pt$ is very small when the periodic time T is 10^{-7} second. When the pressure is 3 millimetres the average value of the force along the axis is about 4·5 volts per centimetre, so that Z/p is about 1·5. The average velocity of the electrons in argon in the direction of the electric force for this value of Z/p is about 10^6 centimetres per second, so that in the time T/2 the electrons will move an average distance of about ·05 centimetre. Thus the electrons do not acquire the amount of energy corresponding to the first critical potential in argon in the time T/2. It is also impossible to account, on this theory, for the fact that the distances between the striations are independent of the frequency of the oscillations.

9. We may here show that the theory of ionization in its simplest form, as given originally by Townsend, affords a rational explanation of the conductivity of uniform luminous columns in discharges through gases.

From a general investigation of the properties of the positive column it was concluded that for small currents the electric force parallel to the axis of the tube was independent of the current, and that the change in the force obtained with large currents may be attributed to the increase in temperature of the gas. It is therefore necessary to find an equation for the coefficient of ionization which is independent of the current.

Since the electric force is independent of the current, the number of electrons q_1 and the number of positive ions q_2 in a cylinder of unit length and of radius r coaxial with the tube are proportional to the current. When the current is steady the rate of increase of these numbers q_1 and q_2 due to ionization by collision is balanced by the rate at which the electrons and positive ions disappear from the gas by flowing to the surface of the tube. The rate of increase of q_1 and q_2 is proportional to q_1, so that the rate at which electrons and positive ions disappear is also proportional to q_1. The disappearance of ions cannot be attributed to recombination in the gas, since the rate of loss due to this process is proportional to $q_1 \times q_2$.

The electrons diffuse to the sides more rapidly than the positive ions, and a positive charge accumulates in the gas due to an excess of the number of positive ions over the number of electrons. The force along the radius due to this positive charge retards the flow of the electrons and increases the flow of the positive ions, and a steady state is reached when q_2 exceeds q_1 by an amount which causes the positive ions and the electrons to flow at the same rate to the surface of the tube. It can thus be seen that the positive charge in the gas is necessary for the maintenance of the steady state *.

By equating the rate at which the electrons and positive ions are generated in the gas to the rate of flow towards the surface of the tube an equation † involving the coefficient of ionization α is obtained which is independent of the current.

The equation reduces to the form

$$C^2 = \alpha W_1^2 / K_1 W_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where K_1 is the coefficient of diffusion of the electrons, W_1 the velocity of the electrons, and W_2 the velocity of the positive ions in the direction of the electric force, and C a constant depending on the radius of the tube and the concentration of positive ions and electrons in the space near the surface of the tube.

The velocity of the positive ions may be expressed in the form $W_2 = a_1 Z/p$, where a_1 is a constant, but the expressions for α, W_1, and K_1 are not so simple. It has been shown that α/p, $K_1 p$, and W_1 are functions of the ratio Z/p ‡, so that equation (1) may be expressed in the form

$$C^2 = p^2 \phi(Z/p). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

This equation gives the value of the force Z in the uniform positive column of a gas at the pressure p. Since this equation has no solution of the form $Z = kp$ where k is a constant, the force Z is not proportional to the pressure.

Equation (1) may be expressed in another form by substituting for K_1 and W_1 the values of these quantities in terms of the mean velocity of agitation $\bar{u}$ of the electrons, assuming the mean free path L of an electron in a gas at unit pressure to be constant. K_1 and W_1 are given by the formulæ $K_1 = a_2 \bar{u}/p$ and $W_1 = a_3 Z/pu$, where a_2 and a_3 are constants.

Also, in steady motion, the mean energy of agitation of the electrons is given by the formula

$$m\bar{U}^2/2 = ZLe/p \sqrt{2\lambda}, \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* J. S. Townsend, 'Electricity in Gases,' Section 302 (1915).

† J. S. Townsend, *Comptes Rendus*, clxxxvi. p. 55 (Jan. 1928).

‡ 'Motion of Electrons in Gases.'

where λ is the proportion of the energy of an electron lost in a collision.

The velocity $\overline{U}$ is therefore proportional to $\sqrt{Z/p}$.

Equation (1) may therefore be written in the form

$$\alpha=\frac{A}{p}\sqrt{Z/p}, \quad \ldots \ldots \quad (4)$$

where A is a constant.

10. From these considerations it appears to us that there are fundamental errors in the hypotheses that have been adopted with regard to the transference of energy in the collisions between electrons and atoms of the gas. It is impossible to admit that there is a high probability of electrons losing their energies in collisions with atoms when their energies are sufficient to excite radiation or ionize the gas.

We notice that other physicists find it difficult to suggest a satisfactory theory of the conductivity in the positive column of discharges on the basis of the generally accepted laws of the interchange of energy between electrons and atoms *.

XXXII. *Distribution of Radiation Resistance in Open Wire Radio Transmission Lines.* *By* T. Walmsley, *B.Sc., M.Inst.C.E.* †

IN considering problems associated with electrical transmission lines for aerials, it is customary to regard the resistance per unit length of line as uniform. Now the resistance includes both ohmic and radiation components, and whilst the former is practically uniform along the line, the latter varies appreciably, being greatest at the ends. It is the purpose of this article to show how the radiation component of resistance varies along a transmission line, and to indicate the numerical value of this component.

An open twin-wire transmission line carrying currents such as are usually obtained when energy is supplied to a short-wave directive aerial array, can be regarded as consisting of two wires having balanced currents, and a single line along which the out-of-balance current flows ‡.

* C. J. Brasefield, Phys. Rev. xxxvii. p. 82 (Jan. 1, 1931).

† Communicated by the Author.

‡ T. Walmsley, "Beam Arrays and Transmission Lines," Journal I. E. E. (Feb. 1931).

Consider such a single line long running at a height $\frac{d}{2}$ above and parallel to the earth. Then the power radiated is the sum of two components—(1) that due to the wire alone, (2) that due to the image,

$$P = P_0 + P_d.$$

Now according to Pistolkers *, the values of P_0 and P_d, computed by the induced electro-motive force method proposed by Brillouin are

$$P_0 = 30 I^2 \left(0{\cdot}577 + \log_e \frac{4\pi l}{\lambda} - Ci \frac{4\pi l}{\lambda}\right),$$

$$P_d = -30 I^2 \left[2 Ci \frac{2\pi d}{\lambda} - Ci \frac{2\pi}{\lambda}(\sqrt{d^2+l^2}+l) - Ci \frac{2\pi}{\lambda}(\sqrt{d^2+l^2}-l)\right],$$

where I is the current, λ the wave-length, and Cix the integral cosine.

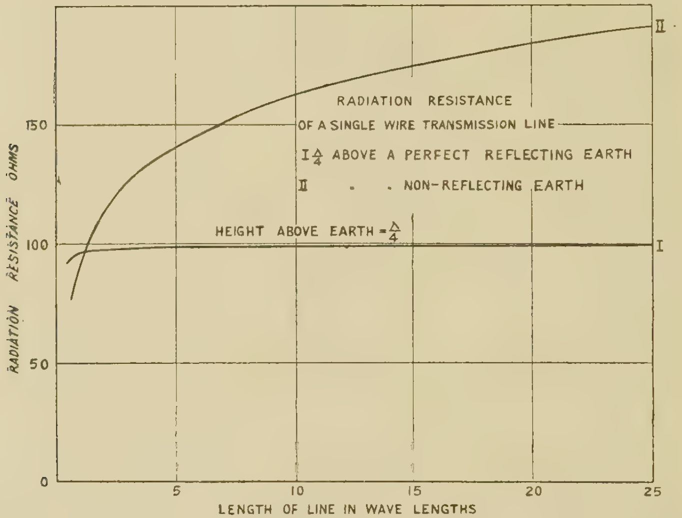
When the earth is regarded as a perfect reflector, the two-power components must be considered, the distance apart of the real and imaginary lines being d; when the earth is considered to be a non-reflector only the component P_0 is concerned, and the height of the line above the earth is immaterial. In fig. 1 the value of the radiation resistance $\frac{P}{I^2}$ has been plotted for a perfectly conducting and a perfectly insulating earth. The curves show that as the length of the transmission line increases the radiation resistance rapidly attains a constant value in the case of a perfectly reflecting earth, and increases at a progressively reduced rate in the case of a non-reflecting earth.

To ascertain the manner in which the radiation resistance of individual sections of the line varies, it is necessary to divide up the line into a series of half-wave sections and calculate the effect of every half-wave element on the particular element concerned. The single equivalent line thus consists of a number of half-wave elements arranged end on, an 180° reversal of phase occurring at points separated a half-wave.

* A. A. Pistolkers, "The Radiation Resistance of Beam Antennas," Proc. Inst. Radio Engrs. xvii. p. 562 (1929).

In the case of a perfectly conducting earth the image of the equivalent line must be taken into account, and can be regarded as a second imaginary line parallel to the equivalent line located at an equal distance below earth as the transmission line is above.

Fig. 1.



The radiation resistance of any element, whether in the imaginary line or the equivalent physical line, is given by the expression *

$$R(d, h) = -15\frac{2\pi h}{\lambda} \left[S\left(d, h-\frac{\lambda}{2}\right) - 2S(d, h) + S\left(d, H+\frac{\lambda}{2}\right)\right],$$

$$-15\cos\frac{2\pi h}{\lambda}\left[C\left(d, h-\frac{\lambda}{2}\right) - 2C(d, h) + C\left(d, h+\frac{\lambda}{2}\right)\right],$$

where $S(x, y)$ and $C(x, y)$ are the functions

$$S(x, y) = \mathrm{Si}\frac{2\pi}{\lambda}(\sqrt{x^2+y^2}+y) - \mathrm{Si}\frac{2\pi}{\lambda}(\sqrt{x^2+y^2}-y),$$

* A. A. Pistolkers, "The Radiation Resistance of Beam Antennas," Proc. Inst. Radio Engrs. xvii. p. 562 (1929).

$$\mathrm{C}(x, y)=\mathrm{Ci}\,\frac{2\pi}{\lambda}\left(\sqrt{x^2+y^2}+y\right)+\mathrm{Ci}\,\frac{2\pi}{\lambda}\left(\sqrt{x^2+y^2}-y\right),$$

Fig. 2.

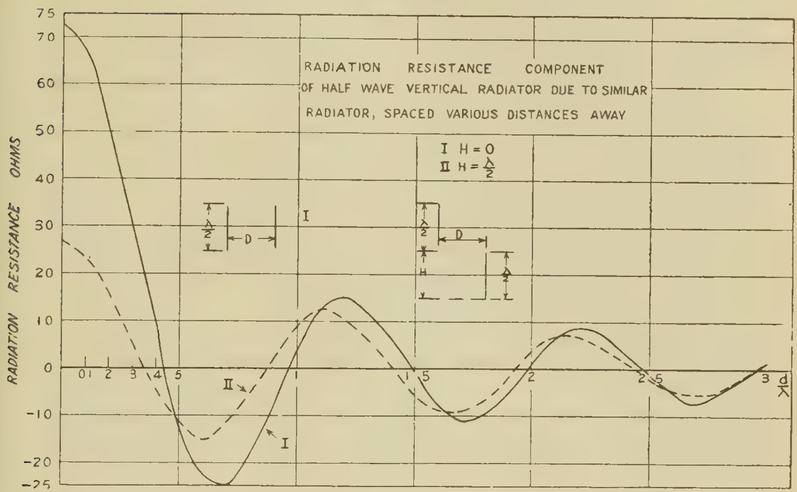
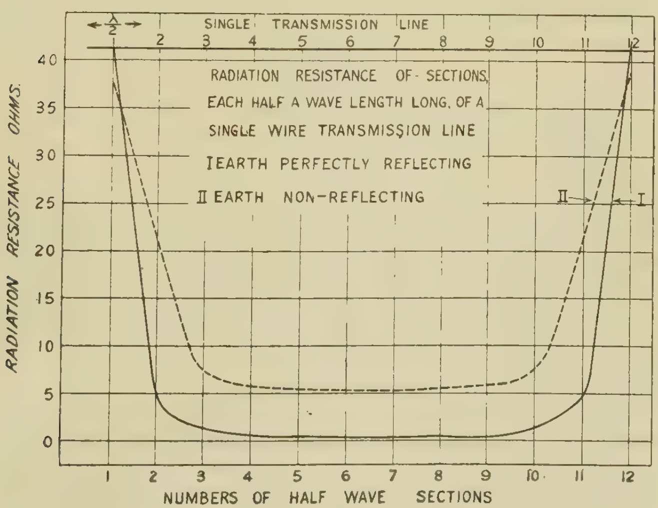


Fig. 3.



and d and h have the significance shown in fig. 2. The very laborious evaluation of this expression has been carried out for values of $h=0$ and $h=\frac{\lambda}{2}$, and is shown in the curves of

fig. 2. Utilizing these curves, a computation has been made of the radiation resistance of sections of a single transmission line 6 wave-lengths long, running above the earth—a height approximating that used for open transmission lines for short-wave directive arrays. The results are shown in fig. 3.

The conclusions to be drawn from the curves of figs. 1 and 3 are, in general, applicable to all transmission lines extending many wave-lengths long. These are, firstly, radiation from end sections is much greater than from other sections of equal length and, secondly, the radiation loss from long lines is not proportionately greater than from short lines. In practice, the amount of out-of-balance current in the single equivalent line of a twin transmission line feeding short-wave arrays depends on the type of array used and need not be great. In these circumstances, the foregoing analyses thus shows that the usual assumption that radiation losses limit the practicable length of transmission lines is incorrect, since short lines have almost as much radiation loss as long lines. The greatest loss in a long line is usually due to high frequency ohmic resistance.

XXXIII. *On the Damping of a Pendulum by Viscous Media.*

By F. E. Hoare, *M.Sc., A.R.C.S.* *

Introduction.

In a former paper † an empirical formula for the rate of decay of the vibration amplitude of a ball pendulum immersed in a viscous fluid was deduced. It was there assumed that the resistance experienced by the sphere arose from two terms, one being proportional to the velocity and the other to the square of the velocity. It is interesting to examine the approximation to the experimental results of the investigation already mentioned, which can be obtained by making use of the various forms which have been proposed for the law of damping in an oscillatory system.

Theoretical Considerations.

The equation of motion for a body performing simple harmonic vibrations with resistance proportional to the velocity can be written

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}+\omega^2x=0,$$

* Communicated by Prof. F. H. Newman, D.Sc.

† Hoare, Phil. Mag. viii. p. 899 (1929).

the solution of which is

$$x = Xe^{-\frac{kt}{2}} \cos(\theta - \gamma).$$

If α_1 and α_2 are any two successive maximum displacements in the same direction we have

$$\frac{\alpha_2}{\alpha_1} = \text{constant},$$

and therefore

$$\alpha_1 - \alpha_2 = A_1 \alpha_1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where A_1 is a constant. As any two observed displacements serve to determine A_1, the theoretical curve of maximum displacement against time can be drawn through any two observed points by the aid of equation (1). This equation is the usual form of solution assumed for vibrating systems, such as galvanometer suspensions etc.

It has been shown by Peirce *, however, that the resistance experienced by such a system cannot always be satisfactorily accounted for on the supposition that it is proportional to the velocity at every instant, the resistance in some cases being much more nearly proportional to the square of the velocity.

Assuming the resistance proportional to the square of the velocity, the equation of motion, if the restoring force is proportional to the displacement, takes the form

$$\frac{d^2x}{dt^2} + k\left(\frac{dx}{dt}\right)^2 + \omega^2 x = 0$$

when the movement is in the positive direction.

A solution of this equation in terms of angular velocity is given by Peirce †, but for our present purposes a solution can best be obtained by writing the equation for the decay of amplitude in the form

$$\frac{d\alpha}{dt} = -a\alpha^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is a constant. Integrating, we have

$$\frac{1}{\alpha} = at + b.$$

If, as before, α_1 and α_2 are any two successive maximum

* Peirce, Proc. Am. Acad. Arts & Sc. xliv. p. 63 (1908).

† Peirce, *loc. cit.*

displacements on the same side, and T the periodic time, we have

$$\frac{1}{\alpha_2} - \frac{1}{\alpha_1} = a\mathrm{T},$$

or, to a very close degree of approximation,

$$\alpha_1 - \alpha_2 = \mathrm{A}_3\alpha_1^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where the periodic time is incorporated in the constant A_3. This equation can be used in the same manner as equation (1) to deduce the theoretical curve for decay of amplitude from any two observed successive maximum displacements. If α_1 and α_2 are only slightly different, it can be seen instantly from the form of equation (2) that the solution is that of (3).

In a similar manner it can be shown that, if the resistance is due in part to a term proportional to the velocity and one proportional to the square of the velocity, the difference of successive amplitudes is given by

$$\alpha_1 - \alpha_2 = \mathrm{A}_4\alpha_1^2 + \mathrm{B}\alpha_1. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Certain experiments of Allen * on the rate of motion of bubbles of gas in rising through a viscous liquid have indicated that when the motion is eddying the resistance experienced by the bubble is proportional to the velocity to the power $\frac{3}{2}$.

If we assume this law of resistance the equation for rate of decay of amplitude becomes

$$\frac{d\alpha}{dt} = -a\alpha^{\frac{3}{2}},$$

which, upon integration and simplification, gives

$$\alpha_1 - \alpha_2 = \mathrm{A}_5\alpha_1^{\frac{3}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In passing, it might be noticed that Allen † found the terminal velocity acquired by large spheres in falling through water could best be explained by a resistance proportional to the square of the velocity, the same result as that found by Newton.

Comparison with Experiment.

In the investigation already referred to spheres of different diameters ($1\frac{1}{2}$, $1\frac{3}{4}$, and 2 inches respectively) were suspended

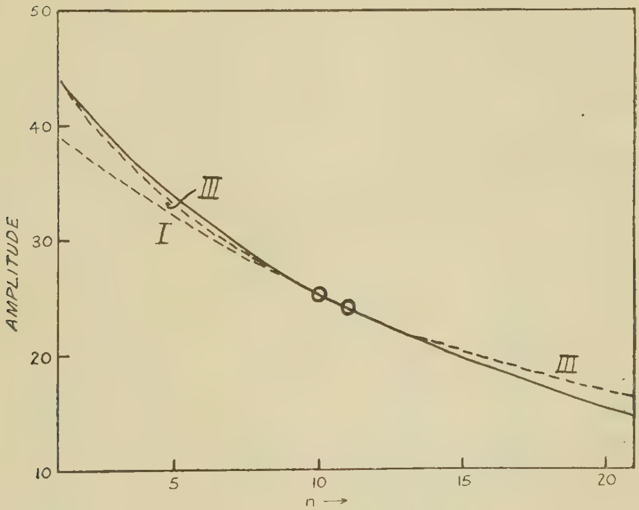
* Allen, Phil. Mag. 1. pp. 325 & 519 (1900).
† Allen, *loc. cit.*

by fine wires and allowed to oscillate in water with periods of approximately 2·5 and 3·5 seconds each. Successive amplitudes, on one side only, were observed for each of the six pendulums.

Equations (1), (3), (4), and (5) have been tested by making use of the results so obtained for the rate of decay of amplitude.

In fig. 1 the full curve represents a typical series of observed results for the 2-inch sphere oscillating with a period of 3·58 seconds. The dotted curves I and III are those given by equations (1) and (3) respectively. These

Fig. 1.



latter have been drawn to pass through two points in the centre of the experimental curve, shown in the figure by the small circles, making use of these two points to determine the constants for the purposes of calculation. In order to work up the curve—that is, to obtain α_1 in terms of α_2—the equations undergo an obvious modification ; this amounts to using the equations in this case in the form

$$\alpha_1 - \alpha_2 = A_1' \alpha_2$$

and

$$\alpha_1 - \alpha_2 = A_3' \alpha_2^2,$$

where the constants A_1' and A_3' differ slightly from A_1 and A_3, but can of course be determined from the same pair of

observations. The observed values of the amplitude are compared with the calculated values in Table I., the observations used to determine the constants being bracketed.

It will be seen from Table I. and fig. 1 that neither equation (1) nor equation (3) is capable of representing the experimental results over the whole range, although the agreement between the latter and those obtained by the application of equation (1) is extremely close for the lower portion of the curve. For the upper portion equation (3) gives a much better approximation, a result in general agreement with that found by Peirce * for certain oscillatory systems.

Table I.

2-inch sphere. Period 3·58 sec.

α observed.	α calculated. Eq. (1)	α calculated. Eq. (3).	α observed.	α calculated. Eq. (1).	α calculated. Eq. (3).
44·0	39·0	44·0	22·9	23·0	23·0
41·0	37·2	40·6	21·8	21·9	22·0
38·3	35·4	37·7	20·9	20·8	21·1
35·8	33·8	35·2	20·0	19·9	20·3
33·9	32·2	33·0	18·9	18·9	19·5
31·9	30·7	31·1	18·0	18·1	18·8
30·1	29·2	29·4	17·1	17·2	18·1
28·2	27·9	27·8	16·2	16·4	17·5
26·6	26·6	26·6	15·4	15·6	16·9
25·3	{25·3}	{25·3}	14·8	14·9	16·4
24·1	{24·1}	{24·1}			

It is somewhat difficult to test equation (4) by making fit observations in the centre of the curve, where the amplitude is decreasing less rapidly, as the two constants cannot then be determined with any great accuracy. Consequently this equation, as well as (5), has been tested by using observations at the commencement to find the constants. The curves given by these equations are shown as IV and V in fig. 2, where the curve I, obtained by using the average value of A_1 given by the first three observations and equation (1), as well as the observed curve, are shown for comparison. The computed amplitudes are given in Table II., the observations used to find the constants being enclosed in brackets in each case.

* Peirce, *loc. cit.*

Table II.

2-inch sphere. Period 3·58 sec.

a observed.	a calculated. Eq. (1).	Eq. (4).	Eq. (5).
44·0	44·0	44·0	44·0
41·0	41·0	41·0	41·0
38·3	38·3	38·3	38·4
35·8	35·8	35·9	35·9
33·9	33·4	33·6	33·7
31·9	31·1	31·6	31·7
30·1	29·0	29·8	29·9
28·2	27·1	28·1	28·2
26·6	25·3	26·5	26·7
25·3	23·6	25·1	25·3
24·1	22·0	23·7	24·0
22·9	20·5	22·5	22·8
21·8	19·2	21·3	21·7
20·9	17·9	20·2	20·7
20·0	16·7	19·2	19·7
18·9	15·6	18.3	18·8
18·0	14·5	17·4	18·0
17·1	13·6	16·6	17·2
16·2	12·6	15·8	16·5
15·4	11·8	15·1	15·8
14·8	11·0	14·4	15·1

Fig. 2.

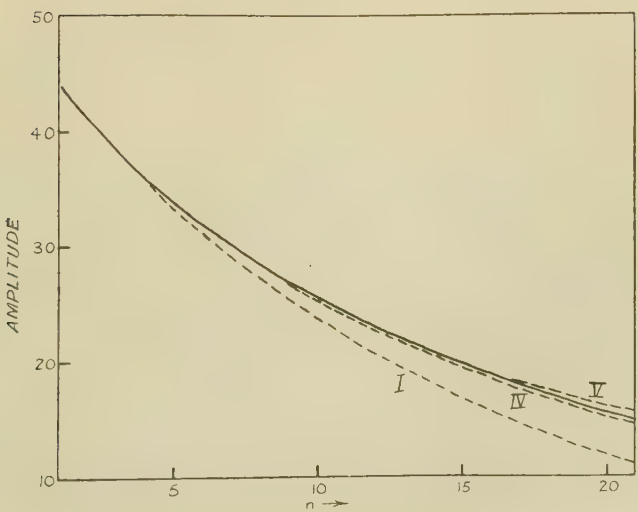
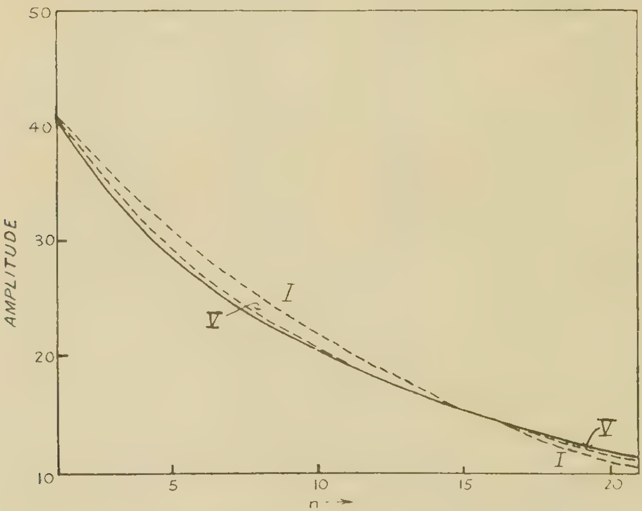


Table II. and fig. 2 show that there is but little to choose between equations (4) and (5) as accurate representations of the experimental results, although the curve given by the latter is on the whole in closer agreement with the observed curve than that given by the former.

In fig. 3 and Table III. are given the results for a $1\frac{1}{2}$-inch ball pendulum oscillating with a period of 2·52 seconds. Owing to the rates of oscillation and decay of amplitude being greater in this case, the experimental results cannot be considered as being of the same accuracy as those given previously. Consequently no attempt has been made to obtain the constants for the various equations from two or

Fig. 3.



three observed amplitudes. The method of least squares has been used instead, in this way making use of all the observations. The agreement between the experimental curve and that given by equation (4) is, in this case, everywhere within the limits of experimental error, so only the curves computed by the use of equations (1) and (5) are shown, in addition to the experimental curve in fig. 3. (The notation used to distinguish the curves is the same as before.)

Discussion of Results.

From the foregoing curves it will be seen that equations (4) and (5) are capable of representing the experimental results with considerable accuracy. Recently Brindley and

Emmerson* have performed somewhat similar experiments with spheres of large diameter oscillating in water, the periodic time being larger than for the results given above. These experimenters found that by confining attention to very small amplitudes the rate of decay of amplitude was in accordance with equation (1). This was not true, however, for the whole range of amplitudes with which they worked, a distinct curvature in the graph of log α against time being

TABLE III.

$1\frac{1}{2}$-inch sphere. Period 2·52 sec.

a observed.	a calculated. Eq. (1).	Eq. (4).	Eq. (5).
41·0	41·0	41·0	41·0
37·0	38·3	37·2	37·5
34·0	35·7	33·9	34·4
31·1	33·4	31·2	31·7
28·8	31·1	28·8	29·3
26·8	29·1	26·7	27·1
24·9	27·1	24·9	25·2
23·2	25·3	23·2	23·5
21·8	23·6	21·7	22·0
20·6	22·1	20·4	20·6
19·4	20·6	19·2	19·3
18·2	19·2	18·1	18·2
17·2	18·0	17·1	17·1
16·3	16·8	16·2	16·2
15·5	15·6	15·4	15·3
14·7	14·6	14·6	14·5
14·0	13·6	13·9	13·8
13·3	12·7	13·3	13·1
12·7	11·9	12·6	12·4
12·1	11·1	12·1	11·8
11·5	10·4	11·5	11·3

apparent at the beginning of the observations, indicating that resistances other than one proportional to the first power of the velocity were acting on the sphere. The effect of a resistance proportional to the square of the velocity will decrease more rapidly than that of one proportional to the first power, and we might reasonably expect that in the experiments of Brindley and Emmerson the point had been reached where the influence of the resistance proportional to the square of the velocity was becoming negligibly small. The elimination of the suspension-wire effect by their method

* Brindley and Emmerson, Phil. Mag. xi. p. 633 (1931).

brings their experimental conditions more into line with those postulated in the development of Stokes's theory—namely, a sphere alone performing simple harmonic vibrations, the velocity always being small. Their results afford satisfactory confirmation of this theory within these limits, but give no adequate guidance as to the law of resistance for larger velocities.

It is suggested by Brindley and Emmerson that the curvature of the log-amplitude curve may be due to the influence of the suspending wire. According to Stokes's theory the wire would also experience a resistance proportional to the velocity, so this would not introduce the observed curvature, unless it is due to some surface effect. The experiment with a sphere half immersed, which was performed by these experimenters, would increase any surface effect, and the curvature should therefore be more pronounced. As no increased curvature in the log-amplitude curve was found it can be concluded that the curvature is not due to any influence of the suspending wire. It should be pointed out, however, that as the theory of damping developed by Stokes is for a sphere totally immersed in a viscous fluid of infinite extent, the interpretation of any results yielded by this experiment with a half-immersed sphere would be a matter of considerable difficulty, for the boundary conditions are different in the two cases.

XXXIV. *Notices respecting New Books.*

Leçons sur le Calcul Vectoriel.
Par T.-A. Ramos. (Blanchard.)

THIS work would provide a convenient introduction to the use of vectors in geometry and physics (though no better, and no worse, than plenty of others) were it not marred by a ridiculous and confusing notation. There is, of course, no unanimity in the notations used for the various types of product of two vectors, the scalar and vector (or inner and outer) products, and Gibbs's "open product"; but this fact scarcely excuses the introduction of an entirely new notation, in which a cross between two vectors indicates the scalar product, and an inverted v, or chevron, the vector product; this is, of course, particularly unhappy in that in one of the commonest notations the vector product is shown by the cross. No mention is made of the open product, or of dyadics or linear vector functions; but these are replaced by a chapter on the tensor calculus, which is not usually found in works of this kind, and is not easy to reconcile in concept and treatment with the rest of the subject-matter. More or less the usual account of the algebra, and differential and integral calculus of vectors, is

given, with applications to the elementary differential geometry of curves and surfaces, to electrical field theory (equally elementary), and so on.

Thermodynamics. By A. W. Porter, F.R.S. (Methuen's Monographs on Physical Subjects. 1931. 2*s*. 6*d*.)

This series of small Monographs has already proved of great value to a wide circle of readers. They aim at giving an up-to-date account of the recent developments in the subjects considered, and are therefore likely to interest not only the University Students and, indeed, the research worker who wishes to know something about subjects allied to his own, but also the large number of those engaged in teaching who are no longer in close contact with recent work.

This small volume gives an interesting outline of Thermodynamics. It discusses the logical foundations of the subject and gives an account of applications of the ideas in various branches of physics. It can be recommended.

Science and First Principles. By F. S. C. Northrop, Associate Professor of Philosophy, Yale University. (Cambridge University Press. 1931. 13*s*. 6*d*.)

This work sets out "to determine precisely what contemporary scientific discoveries in many different branches of science reveal, and what all this means for philosophy." So wide an aim might well daunt the scientific worker, who would but rarely consider himself competent to appraise the achievements in more than one or two branches of science. This volume shows no such timidity and ranges from Greek science to the Principle of Relativity, from Hegel to Dirac, from Aristotle to Haldane, and from Hume to Whitehead and Russell. To the philosopher, standing outside modern developments and scientific work, such a work may be of value. And it is anyway interesting that an attempt has been made to unify such very dissimilar lines of work.

XXXV. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 200.]

May 6th, 1931.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S., President, in the Chair.

THE following communications were read:—

1. 'The Geology of the Country around Mynydd Rhiw and Sarn, South-Western Lleyn (Carnarvonshire).' By Charles Alfred Matley, D.Sc., F.G.S.

The region described contains about 9 square miles of the Palæozoic country of South-Western Lleyn. It is situated between

Porth Neigwl on the east and the Pre-Cambrian rocks of the Mona Complex on the west, and includes a ridge of high ground which stretches from Mynydd Penarfynydd in the south to Mynydd Cefnamwlch in the north, and attains a height of 994 feet on Mynydd Rhiw. Much of the lower ground is covered by glacial drift.

The sedimentary rocks are of Arenig and Lower Llanvirn age, and range from the Extensus Zone to a high part of the Bifidus Zone. The local base of the Arenig is found at only one small exposure, and there it rests on the mica-schists (Penmynydd Zone) of the Mona Complex.

In the Hirundo Zone, and extending into the Bifidus Zone, is the Rhiw Volcanic Group, in which four lava-flows have been found, as well as rhyolitic ashes and ashy sediments. The manganese-ore of Nant y Gadwen and Rhiw is a metasomatized ash, and has been taken as the base of this group. Three of the flows at Rhiw are spilitic, ranging upwards from albite-basalt to spilite. The other is a purple rhyolite. The lower lavas at Rhiw, and the one at Sarn described by Prof. O. T. Jones, have pillow structure in the basal part only, the remainder of each flow often showing good columnar structure. Comparison is made with some Merioneth lavas described by A. H. Cox and A. K. Wells, in which the same combination of structures occurs, but in which the relative position in the flows of pillow and columnar structure is reversed.

Intrusive rocks are abundant, and range in composition from acidic to ultra-basic; they are linked genetically by their common richness in soda, and can be considered, with the volcanic rocks, as varied members of one spilitic suite. They include the Sarn Granite, numerous albite-dolerite sills, and the coarser-grained intrusions of picrite, proterobase, and hornblende-dolerite which are found only at the top of the sequence. These intrusions are discussed, and the possibility considered that they may be a single sill or laccolith broken and displaced by faulting. All the igneous rocks, extrusive and intrusive, thin out northwards.

A few albite-dolerite dykes of Palæozoic age have been found, two of which intrude into the Sarn Granite. A perlitic soda-rhyolite dyke, found near the Sarn Granite, seems to be an offshoot from it.

A correlation is made of the vulcanicity of this area with that in other parts of Wales and Shropshire. The paper also contains some remarks on the tectonics.

2. 'An Ordovician Grit from Anglesey, with its bearings upon Palæogeography, and upon the Tectonics of the Mona Complex.' By Edward Greenly, D.Sc., F.G.S., and Prof. Percy George Hamnall Boswell, D.Sc., F.R.S., F.G.S.

The grits at the base of the Arenig beds at Berw, near Holland Arms, have been found to be exceptionally rich in heavy minerals.

Garnet, sphene, and epidote are wonderfully abundant, as are ilmenite, biotite, and white mica. The felspars are orthoclase, albite, and oligoclase. Tourmaline and glaucophane are absent, and only one grain of green hornblende has been seen. There are also small pebbles of Penmynydd Zone mica-schist, Gwna green-schist, and many of granitoid acid gneiss.

The source cannot be far distant, for the deposits are poorly graded, and the grains angular or even splintery. A source near at hand, however, can be only the Mona Complex.

The absence of tourmaline excludes at once the middle region, so the orthoclase must be traced to the augen of the Penmynydd Zone. The albite, oligoclase, garnet, sphene, ilmenite, and other minerals can only have come from the gneisses. The absence of amphiboles excludes both the basic gneisses and the basic members of the Penmynydd Zone, in spite of the proximity of those rocks to Berw. Accordingly, the evidence both of minerals and rock-fragments points to the acid gneisses as the principal source, with contributions from the Gwna Beds and the acid members of the Penmynydd Zone.

The basic members of the Penmynydd Zone would have been, in Ordovician time, covered by Gwna Beds; not so the basic gneisses, for they are not affected by the Gwna metamorphism. Although large masses of acid gneiss, now invisible, have been invoked, they cannot lie buried under the coalfield, for the slopes of the sub-Ordovician land all converged towards the Trewan cirque, and the slope at Berw must, therefore, have been westward. This acid gneiss must have lain higher up, and have been destroyed by waste. In that case it would provide the needed shield for the basic gneiss.

Finally, this furnishes unexpected confirmation of a tentative hypothesis, already put forward, that the gneiss of Holland Arms, instead of having been brought up from below, has been brought down from above, on an infold of the Newborough slide, from the inverted upper limb of the Bodorgan recumbent fold.

May 20th, 1931.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S., President, in the Chair.

The following communication was read :—

'The Geology of the Country between Ivybridge and Buckfastleigh, Devon.' By Albert Alfred Fitch, B.Sc., A.R.C.S., F.G.S.

The area described consists of a strip of the granite margin, the metamorphic aureole, and some rocks beyond the influence of the granite.

The physical geology is discussed, and the presence of a relic of the 700–750-foot platform demonstrated by planimetric measurements.

Stratigraphically, the rocks comprise

Carboniferous	D 3		Culm, with a sandstone band and an intrusive dolerite.
Devonian	Frasnian		Slates. Limestone.
	Givetian		Spilitic complex. Slates.

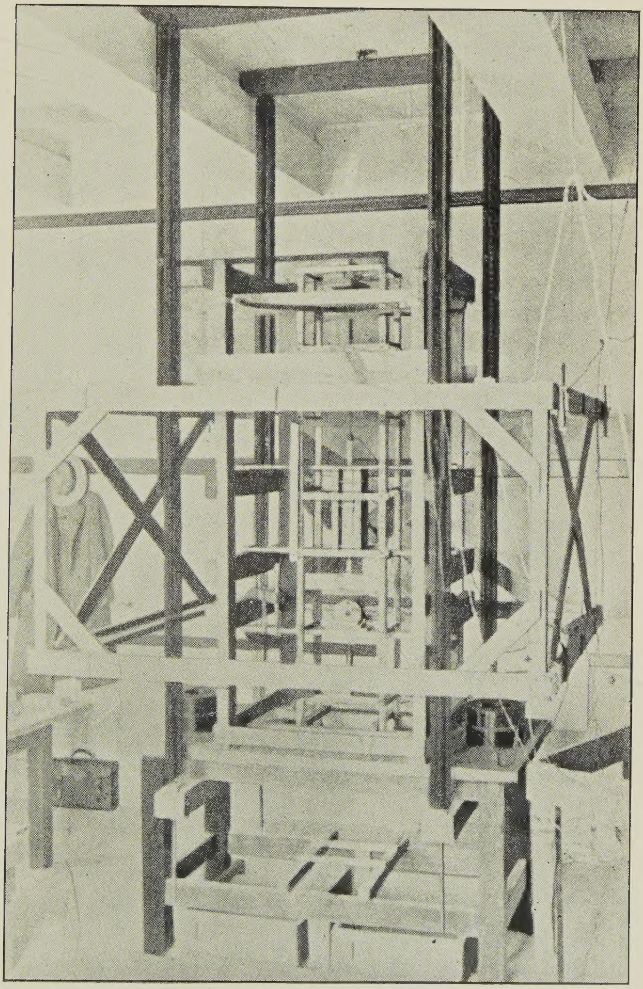
The Devonian is folded into a syncline, while the Culm is present only in a trough-faulted block, which appears to include a slice across the southern margin of the granite, with the Culm included at both ends, where the two faults converge. The sequence and structure are cut across by the Dartmoor Granite, and the several rock-types undergo a varied suite of changes, controlled by the degree of thermal metamorphism, of pneumatolytic and hydrothermal action, and the original lithology. The petrology and petrogenesis of the resulting rocks are discussed in detail, and some aspects of two-way migration considered.

The mineral assemblage of the aureole is not of much diagnostic value for the provenance of the sediments of the South of England. Superficial deposits do not afford matter of great interest.

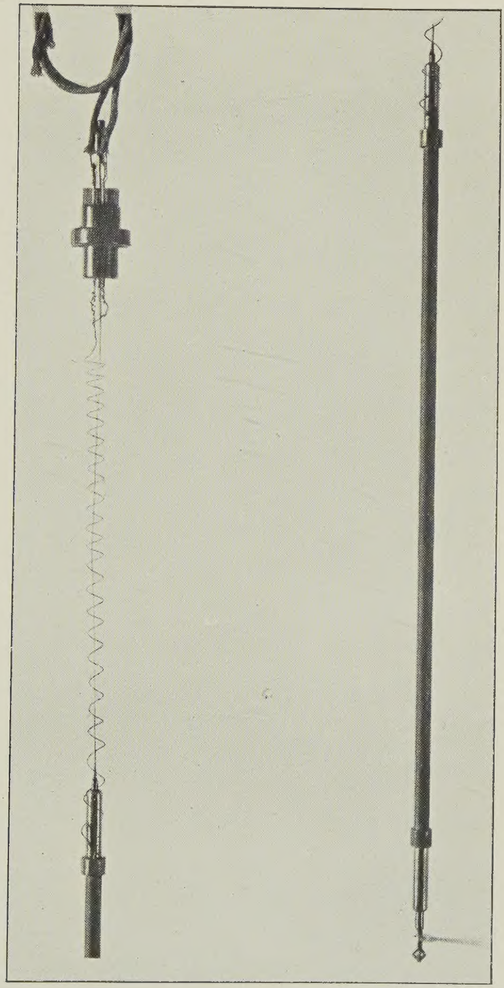
Economic aspects of the geology are dealt with.

The SECRETARY (Mr. W. CAMPBELL SMITH) exhibited a series of specimens from the contact of the quartz-dolerite of the Newark Group with the Brunswick Shale (Trias), sent to the Geological Society by Dr. Henry Darwin Rogers about 1840 and transferred to the British Museum (Natural History) in 1911. The specimens came from New Hope, Bucks County, Pennsylvania, on the Delaware River. The altered rocks contain large idiomorphic black tourmalines in a biotite-hornfels, and the demonstration of the presence of tourmaline at a dolerite contact was the chief point of the exhibit. Tourmaline was not recorded at New Hope until 1889, although H. D. Rogers himself had described the occurrence of identical rocks on the New Jersey side of the Delaware River in 1836. The idocrase associated with epidote described by Rogers from New Hope appear to be zoisite. Contact metamorphic phenomena similar to those seen in these specimens were fully described by J. Volney Lewis in 1908 (Geol. Surv. New Jersey, Ann. Rep. State Geologist for 1907, 1908, pp. 138–47).

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]



The apparatus near the rotor



The glass rotor, with upper suspension and torsion-head.